Rotational Motion

7.1 Introduction

Translation is motion along a straight line but rotation is the motion of wheels, gears, motors, planets, the hands of a clock, the rotor of jet engines and the blades of helicopters. First figure shows a skater gliding across the ice in a straight line with constant speed. Her motion is called translation but second figure shows her spinning at a constant rate about a vertical axis. Here motion is called rotation.

Up to now we have studied translatory motion of a point mass. In this chapter we will study the rotatory motion of rigid body about a fixed axis.

(1) Rigid body : A rigid body is a body that can rotate with all the parts locked together and without any change in its shape.

(2) System : A collection of any number of particles interacting with one another and are under consideration during analysis of a situation are said to form a system.

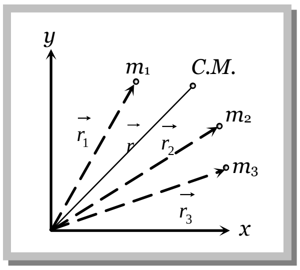
(3) Internal forces : All the forces exerted by various particles of the system on one another are called internal forces. These forces are alone enable the particles to form a well defined system. Internal forces between two particles are mutual (equal and opposite).

(4) External forces : To move or stop an object of finite size, we have to apply a force on the object from outside. This force exerted on a given system is called an external force.

7.2 Centre of Mass

Centre of mass of a system (body) is a point that moves as though all the mass were concentrated there and all external forces were applied there.

(1) **Position vector of centre of mass for *n* particle system :** If a system consists of *n* particles of masses $m_1, m_2, m_3 ...... m_n$, whose positions vectors are $\vec{r}_1, \vec{r}_2, \vec{r}_3 .... \vec{r}_n$ respectively then position vector of centre of mass



$$\vec{r} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + ... m_n \vec{r}_n}{m_1 + m_2 + m_3 + ... m_n}$$

Hence the centre of mass of *n* particles is a weighted average of the position vectors of *n* particles making up the system.

(2) **Position vector of centre of mass for two particle system :** $\vec{r} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$

and the centre of mass lies between the particles on the line joining them.

If two masses are equal *i.e.* $m_1 = m_2$, then position vector of centre of mass $\vec{r} = \dfrac{\vec{r}_1 + \vec{r}_2}{2}$

(3) **Important points about centre of mass**

(i) The position of centre of mass is independent of the co-ordinate system chosen.

(ii) The position of centre of mass depends upon the shape of the body and distribution of mass.

Example : The centre of mass of a circular disc is within the material of the body while that of a circular ring is outside the material of the body.

(iii) In symmetrical bodies in which the distribution of mass is homogenous, the centre of mass coincides with the geometrical centre or centre of symmetry of the body.

(iv) Position of centre of mass for different bodies

S. No.	**Body**	**Position of centre of mass**
(a)	Uniform hollow sphere	Centre of sphere
(b)	Uniform solid sphere	Centre of sphere
(c)	Uniform circular ring	Centre of ring
(d)	Uniform circular disc	Centre of disc
(e)	Uniform rod	Centre of rod
(f)	A plane lamina (Square, Rectangle, Parallelogram)	Point of inter section of diagonals
(g)	Triangular plane lamina	Point of inter section of medians
(h)	Rectangular or cubical block	Points of inter section of diagonals
(i)	Hollow cylinder	Middle point of the axis of cylinder
(j)	Solid cylinder	Middle point of the axis of cylinder
(k)	Cone or pyramid	On the axis of the cone at point distance $\frac{3h}{4}$ from the vertex where h is the height of cone

(v) The centre of mass changes its position only under the translatory motion. There is no effect of rotatory motion on centre of mass of the body.

(vi) If the origin is at the centre of mass, then the sum of the moments of the masses of the system about the centre of mass is zero *i.e.* $\Sigma m_i \vec{r_i} = 0$.

(vii) If a system of particles of masses $m_1, m_2, m_3, ...$ move with velocities $v_1, v_2, v_3, ...$

then the velocity of centre of mass $v_{cm} = \frac{\Sigma m_i v_i}{\Sigma m_i}$.

(viii) If a system of particles of masses $m_1, m_2, m_3, ...$ move with accelerations $a_1, a_2, a_3, ...$

then the acceleration of centre of mass $A_{cm} = \frac{\Sigma m_i a_i}{\Sigma m_i}$

(ix) If $\vec{r}$ is a position vector of centre of mass of a system

then velocity of centre of mass $\vec{v}_{cm} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\frac{m_1 \vec{r_1} + m_2 \vec{r_2} + m_3 \vec{r}_3 + ...}{m_1 + m_2 + m_3 + ...}\right)$

(x) Acceleration of centre of mass $\vec{A}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2}{dt^2}\left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ...}{m_1 + m_2 + m_3 + ...}\right)$

(xi) Force on a rigid body $\vec{F} = M\vec{A}_{cm} = M\frac{d^2\vec{r}}{dt^2}$

(xii) For an isolated system external force on the body is zero

$$\vec{F} = M\frac{d}{dt}\left(\vec{v}_{cm}\right) = 0 \Rightarrow \vec{v}_{cm} = \text{constant.}$$

i.e., centre of mass of an isolated system moves with uniform velocity along a straight-line path.

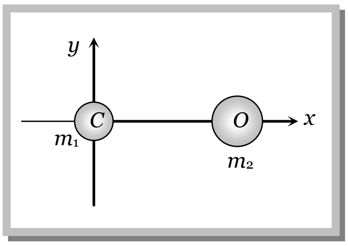
Sample problems based on centre of mass

Problem 1. The distance between the carbon atom and the oxygen atom in a carbon monoxide molecule is 1.1 Å. Given, mass of carbon atom is 12 *a.m.u.* and mass of oxygen atom is 16 *a.m.u.*, calculate the position of the center of mass of the carbon monoxide molecule

(a) 6.3 Å from the carbon atom (b) 1 Å from the oxygen atom

(c) 0.63 Å from the carbon atom (d) 0.12 Å from the oxygen atom

Solution : (c) Let carbon atom is at the origin and the oxygen atom is placed at *x*-axis



$m_1 = 12$, $m_2 = 16$, $\vec{r}_1 = 0\hat{i} + 0\hat{j}$ and $\vec{r}_2 = 1.1\hat{i} + 0\hat{j}$

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{16 \times 1.1}{28}\hat{i}$$

$\vec{r} = 0.63\hat{i}$ *i.e.* 0.63 Å from carbon atom.

Problem 2. The velocities of three particles of masses 20*g*, 30*g* and 50 *g* are $10\vec{i}, 10\vec{j}$, and $10\vec{k}$ respectively. The velocity of the centre of mass of the three particles is **[EAMCET 2001]**

(a) $2\vec{i} + 3\vec{j} + 5\vec{k}$ (b) $10(\vec{i} + \vec{j} + \vec{k})$ (c) $20\vec{i} + 30\vec{j} + 5\vec{k}$ (d) $2\vec{i} + 30\vec{j} + 50\vec{k}$

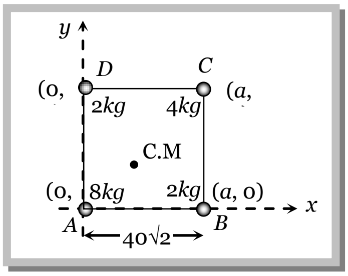
Solution : (a) Velocity of centre of mass

$$v_{cm} = \frac{m_1v_1 + m_2v_2 + m_3v_3}{m_1 + m_2 + m_3} = \frac{20 \times 10\hat{i} + 30 \times 10\hat{j} + 50 \times 10\hat{k}}{100} = 2\hat{i} + 3\hat{j} + 5\hat{k}.$$

Problem 3. Masses 8, 2, 4, 2 *kg* are placed at the corners *A, B, C, D* respectively of a square *ABCD* of diagonal 80 *cm*. The distance of centre of mass from *A* will be

(a) 20 *cm* (b) 30 *cm* (c) 40 *cm* (d) 60 *cm*

Solution : (b) Let corner A of square *ABCD* is at the origin and the mass 8 *kg* is placed at this corner (given in problem) Diagonal of square $d = a\sqrt{2} = 80\ cm \Rightarrow a = 40\sqrt{2}cm$



$m_1 = 8kg,\ m_2 = 2kg,\ m_3 = 4kg,\ m_4 = 2kg$

Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$ are the position vectors of respective masses

$$\vec{r}_1 = 0\hat{i} + 0\hat{j},\ \vec{r}_2 = a\hat{i} + 0\hat{j},\ \vec{r}_3 = a\hat{i} + a\hat{j},\ \vec{r}_4 = 0\hat{i} + a\hat{j}$$

From the formula of centre of mass

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4}{m_1 + m_2 + m_3 + m_4} = 15\sqrt{2}i + 15\sqrt{2}\hat{j}$$

$\therefore$ co-ordinates of centre of mass $= (15\sqrt{2}, 15\sqrt{2})$ and co-ordination of the corner $= (0, 0)$

From the formula of distance between two points (x_1, y_1) and (x_2, y_2)

distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(15\sqrt{2} - 0)^2 + (15\sqrt{2} - 0)^2} = \sqrt{900} = 30cm$

***Problem* 4.** The coordinates of the positions of particles of mass $7, 4$ and $10\ gm$ are $(1, 5, -3)$, $(2, 5, 7)$ and $(3, 3, -1)\,cm$ respectively. The position of the centre of mass of the system would be

(a) $\left(-\frac{15}{7}, \frac{85}{17}, \frac{1}{7}\right) cm$ (b) $\left(\frac{15}{7}, -\frac{85}{17}, \frac{1}{7}\right) cm$ (c) $\left(\frac{15}{7}, \frac{85}{21}, -\frac{1}{7}\right) cm$ (d) $\left(\frac{15}{7}, \frac{85}{21}, \frac{7}{3}\right) cm$

Solution: (c) $m_1 = 7gm$, $m_2 = 4gm$, $m_3 = 10gm$ and $\vec{r_1} = (\hat{i} + 5\hat{j} - 3\hat{k})$, $r_2 = (2i + 5j + 7k)$, $r_3 = (3\hat{i} + 3\hat{j} - \hat{k})$

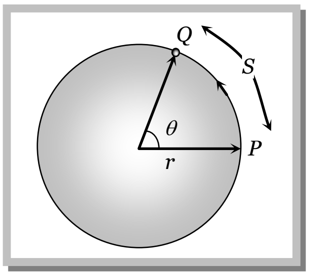
Position vector of center mass

$$\vec{r} = \frac{7(\hat{i} + 5\hat{j} - 3\hat{k}) + 4(2\hat{i} + 5\hat{j} + 7\hat{k}) + 10(3\hat{i} + 3\hat{j} - \hat{k})}{7 + 4 + 10} = \frac{(45\hat{i} + 85\hat{j} - 3\hat{k})}{21}$$

$\Rightarrow \vec{r} = \frac{15}{7}\hat{i} + \frac{85}{21}\hat{j} - \frac{1}{7}\hat{k}$. So coordinates of centre of mass $\left[\frac{15}{7}, \frac{85}{21}, \frac{-1}{7}\right]$.

7.3 Angular Displacement

It is the angle described by the position vector $\vec{r}$ about the axis of rotation.



Angular displacement $(\theta) = \frac{\text{Linear displacement } (s)}{\text{Radius } (r)}$

(1) Unit : radian

(2) Dimension : $[M^0L^0T^0]$

(3) Vector form $\vec{S} = \vec{\theta} \times \vec{r}$

i.e., angular displacement is a vector quantity whose direction is given by right hand rule. It is also known as axial vector. For anti-clockwise sense of rotation direction of θ is perpendicular to the plane, outward and along the axis of rotation and vice-versa.

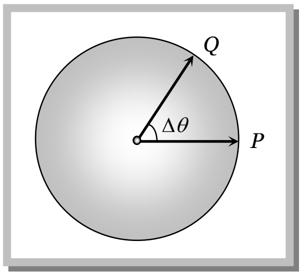
(4) 2π radian $= 360° = 1$ revolution.

(5) If a body rotates about a fixed axis then all the particles will have same angular displacement (although linear displacement will differ from particle to particle in accordance with the distance of particles from the axis of rotation).

7.4 Angular Velocity

The angular displacement per unit time is defined as angular velocity.

If a particle moves from P to Q in time Δt, $\omega = \frac{\Delta\theta}{\Delta t}$ where $\Delta\theta$ is the angular displacement.



(1) Instantaneous angular velocity $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$

(2) Average angular velocity $\omega_{av} = \frac{\text{total angular displacement}}{\text{total time}} = \frac{\theta_2 - \theta_1}{t_2 - t_1}$

(3) Unit : *Radian/sec*

(4) Dimension : $[M^0L^0T^{-1}]$ which is same as that of frequency.

(5) Vector form $\vec{v} = \vec{\omega} \times \vec{r}$ [where $\vec{v}$ = linear velocity, $\vec{r}$ = radius vector]

$\vec{\omega}$ is a axial vector, whose direction is normal to the rotational plane and its direction is given by right hand screw rule.

(6) $\omega = \frac{2\pi}{T} = 2\pi n$ [where T = time period, n = frequency]

(7) The magnitude of an angular velocity is called the angular speed which is also represented by ω .

7.5 Angular Acceleration

The rate of change of angular velocity is defined as angular acceleration.

If particle has angular velocity ω_1 at time t_1 and angular velocity ω_2 at time t_2 then,

Angular acceleration $\vec{\alpha} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1}$

(1) Instantaneous angular acceleration $\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\vec{\omega}}{dt} = \frac{d^2 \vec{\theta}}{dt^2}$.

(2) Unit : rad/sec^2

(3) Dimension : $[M^0 L^0 T^{-2}]$.

(4) If $\alpha = 0$, circular or rotational motion is said to be uniform.

(5) Average angular acceleration $\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$.

(6) Relation between angular acceleration and linear acceleration $\vec{a} = \vec{\alpha} \times \vec{r}$.

(7) It is an axial vector whose direction is along the change in direction of angular velocity *i.e.* normal to the rotational plane, outward or inward along the axis of rotation (depends upon the sense of rotation).

7.6 Equations of Linear Motion and Rotational Motion

	Linear Motion	**Rotational Motion**
(1)	If linear acceleration is 0, u = constant and $s = u\,t$.	If angular acceleration is 0, ω = constant and $\theta = \omega t$
(2)	If linear acceleration a = constant, (i) $s = \frac{(u+v)}{2} t$ (ii) $a = \frac{v-u}{t}$ (iii) $v = u + at$ (iv) $s = ut + \frac{1}{2} at^2$ (v) $v^2 = u^2 + 2as$ (vi) $s_{nth} = u + \frac{1}{2} a(2n-1)$	If angular acceleration α = constant then (i) $\theta = \frac{(\omega_1 + \omega_2)}{2} t$ (ii) $\alpha = \frac{\omega_2 - \omega_1}{t}$ (iii) $\omega_2 = \omega_1 + \alpha t$ (iv) $\theta = \omega_1 t + \frac{1}{2} \alpha t^2$ (v) $\omega_2^2 = \omega_1^2 + 2\alpha\theta$ (vi) $\theta_{nth} = \omega_1 + (2n-1)\frac{\alpha}{2}$
(3)	If acceleration is not constant, the above equation will not be applicable. In this case	If acceleration is not constant, the above equation will not be applicable. In this case

(i) $v = \frac{dx}{dt}$	(i) $\omega = \frac{d\theta}{dt}$
(ii) $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	(ii) $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
(iii) $vdv = a\,ds$	(iii) $\omega d\omega = \alpha\, d\theta$

Sample problems based on angular displacement, velocity and acceleration

Problem 5. The angular velocity of *seconds* hand of a watch will be

(a) $\frac{\pi}{60} rad / sec$ (b) $\frac{\pi}{30} rad / sec$ (c) $60\pi\ rad / sec$ (d) $30\pi\ rad / sec$

Solution : (b) We know that second's hand completes its revolution (2π) in 60 *sec* $\therefore\ \omega = \frac{\theta}{t} = \frac{2\pi}{60} = \frac{\pi}{30} rad/sec$

Problem 6. The wheel of a car is rotating at the rate of 1200 revolutions per *minute*. On pressing the accelerator for 10 *sec* it starts rotating at 4500 revolutions per *minute*. The angular acceleration of the wheel is **[MP PET 2001]**

(a) 30 *radians/sec*^{2} (b) 1880 *degrees/sec*^{2} (c) 40 *radians/sec*^{2} (d) 1980 *degrees/sec*^{2}

Solution: (d) Angular acceleration (α) = rate of change of angular speed

$$= \frac{2\pi(n_2 - n_1)}{t} = \frac{2\pi\left(\frac{4500 - 1200}{60}\right)}{10} = \frac{2\pi\frac{3300}{60}}{10} \times \frac{360}{2\pi}\frac{degree}{sec^2} = 1980\ degree / sec^2.$$

Problem 7. Angular displacement (θ) of a flywheel varies with time as $\theta = at + bt^2 + ct^3$ then angular acceleration is given by

(a) $a + 2bt - 3ct^2$ (b) $2b - 6t$ (c) $a + 2b - 6t$ (d) $2b + 6ct$

Solution: (d) Angular acceleration $\alpha = \frac{d^2\theta}{dt^2} = \frac{d^2}{dt^2}(at + bt^2 + ct^3) = 2b + 6ct$

Problem 8. A wheel completes 2000 rotations in covering a distance of $9.5\ km$. The diameter of the wheel is **[RPMT 199**

(a) $1.5\,m$ (b) $1.5\,cm$ (c) $7.5\,m$ (d) $7.5\,cm$

Solution: (a) Distance covered by wheel in 1 rotation = $2\pi r = \pi D$ (Where *D*= 2*r* = diameter of wheel)

$\therefore$ Distance covered in 2000 rotation = 2000 πD = $9.5 \times 10^3 m$ (given)

$\therefore D = 1.5\ meter$

Problem 9. A wheel is at rest. Its angular velocity increases uniformly and becomes 60 *rad/sec* after 5 *sec*. The total angular displacement is

(a) 600 *rad* (b) 75 *rad* (c) 300 *rad* (d) 150 *rad*

Solution: (d) Angular acceleration $\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{60 - 0}{5} = 12 rad / sec^2$

Now from $\theta = \omega_1 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}(12)(5)^2 = 150\ rad.$

Problem 10. A wheel initially at rest, is rotated with a uniform angular acceleration. The wheel rotates through an angle θ_1 in first one second and through an additional angle θ_2 in the next one second. The ratio $\frac{\theta_2}{\theta_1}$ is

(a) 4 (b) 2 (c) 3 (d) 1

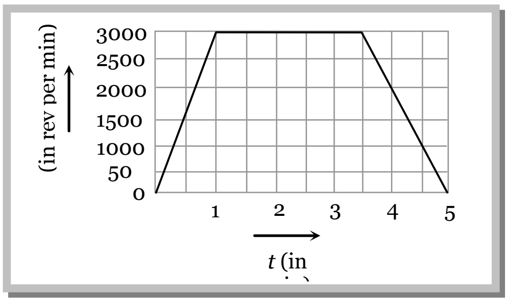
Solution: (c) Angular displacement in first one second $\theta_1 = \frac{1}{2}\alpha(1)^2 = \frac{\alpha}{2}$ ......(i) [From $\theta = \omega_1 t + \frac{1}{2}\alpha t^2$]

Now again we will consider motion from the rest and angular displacement in total two seconds

$$\theta_1 + \theta_2 = \frac{1}{2}\alpha(2)^2 = 2\alpha \quad ......(ii)$$

Solving (i) and (ii) we get $\theta_1 = \frac{\alpha}{2}$ and $\theta_2 = \frac{3\alpha}{2}$ $\therefore \frac{\theta_2}{\theta_1} = 3$.

***Problem* 11.** As a part of a maintenance inspection the compressor of a jet engine is made to spin according to the graph as shown. The number of revolutions made by the compressor during the test is

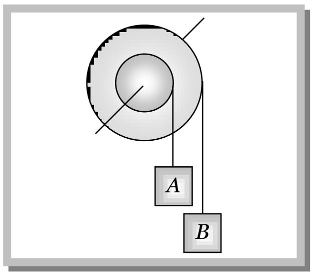


(a) 9000 (b) 16570 (c) 12750 (d) 11250

Solution: (d) Number of revolution = Area between the graph and time axis = Area of trapezium

$$= \frac{1}{2}\times(2.5+5)\times 3000 = 11250 \text{ revolution.}$$

***Problem* 12.** Figure shows a small wheel fixed coaxially on a bigger one of double the radius. The system rotates about the common axis. The strings supporting *A* and *B* do not slip on the wheels. If *x* and *y* be the distances travelled by *A* and *B* in the same time interval, then



(a) $x = 2y$

(b) $x = y$

(c) $y = 2x$

(d) None of these

Solution: (c) Linear displacement (*S*) = Radius (*r*) × Angular displacement (θ)

$\therefore S \propto r$ (if $\theta =$ constant)

$$\frac{\text{Distance travelled by mass } A(x)}{\text{Distance travelled by mass } B(y)} = \frac{\text{Radius of pulley concerned with mass } A(r)}{\text{Radius of pulley concerned with mass } B(2r)} = \frac{1}{2} \Rightarrow y = 2x.$$

***Problem* 13.** If the position vector of a particle is $\vec{r} = (3\hat{i} + 4\hat{j})$ meter and its angular velocity is $\vec{\omega} = (\hat{j} + 2\hat{k})$ *rad/sec* then its linear velocity is (in *m/s*)

(a) $(8\hat{i} - 6\hat{j} + 3\hat{k})$ (b) $(3\hat{i} + 6\hat{j} + 8\hat{k})$ (c) $-(3\hat{i} + 6\hat{j} + 6\hat{k})$ (d) $(6\hat{i} + 8\hat{j} + 3\hat{k})$

Solution: (a) $\vec{v} = \vec{\omega} \times \vec{r} = (3\hat{i} + 4\hat{j} + 0\hat{k}) \times (0\hat{i} + \hat{j} + 2\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 8\hat{i} - 6\hat{j} + 3\hat{k}$

7.7 Moment of Inertia

Moment of inertia plays the same role in rotational motion as mass plays in linear motion. It is the property of a body due to which it opposes any change in its state of rest or of uniform rotation.

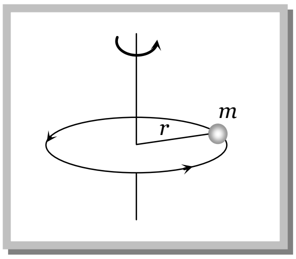
(1) Moment of inertia of a particle $I = mr^2$; where r is the perpendicular distance of particle from rotational axis.

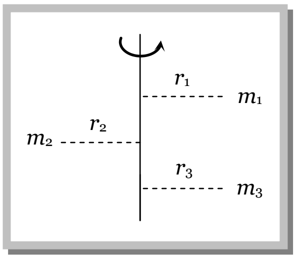
(2) Moment of inertia of a body made up of number of particles (discrete distribution)

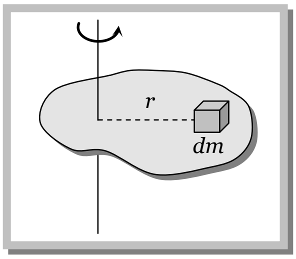
$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...$$

(3) Moment of inertia of a continuous distribution of mass, treating the element of mass dm at position r as particle

$$dI = dm\, r^2 \quad i.e.,\ I = \int r^2 dm$$







(4) Dimension : $[ML^2T^0]$

(5) S.I. unit : kgm^2.

(6) Moment of inertia depends on mass, distribution of mass and on the position of axis of rotation.

(7) Moment of inertia does not depend on angular velocity, angular acceleration, torque, angular momentum and rotational kinetic energy.

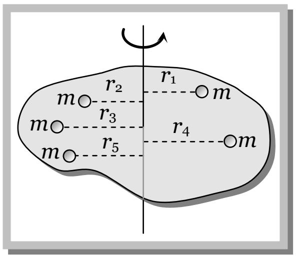
(8) It is not a vector as direction (clockwise or anti-clockwise) is not to be specified and also not a scalar as it has different values in different directions. Actually it is a tensor quantity.

(9) In case of a hollow and solid body of same mass, radius and shape for a given axis, moment of inertia of hollow body is greater than that for the solid body because it depends upon the mass distribution.

7.8 Radius of Gyration

Radius of gyration of a body about a given axis is the perpendicular distance of a point from the axis, where if whole mass of the body were concentrated, the body shall have the same moment of inertia as it has with the actual distribution of mass.

When square of radius of gyration is multiplied with the mass of the body gives the moment of inertia of the body about the given axis.



$$I = Mk^2 \text{ or } k = \sqrt{\frac{I}{M}}.$$

Here k is called radius of gyration.

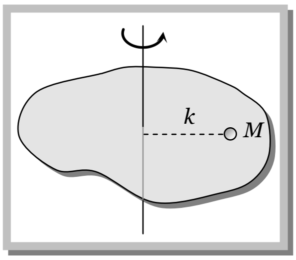
From the formula of discrete distribution

$$I = mr_1^2 + mr_2^2 + mr_3^2 + ... + mr_n^2$$

If $m_1 = m_2 = m_3 = ... = m$ then

$$I = m(r_1^2 + r_2^2 + r_3^2 + ... r_n^2) \qquad ....(i)$$

From the definition of Radius of gyration,



$$I = Mk^2 \qquad ....(ii)$$

By equating (i) and (ii)

$$Mk^2 = m(r_1^2 + r_2^2 + r_3^2 + \dots\dots\dots .. + r_n^2)$$

$$nmk^2 = m(r_1^2 + r_2^2 + r_3^2 + \dots\dots\dots + r_n^2) \qquad [\text{As } M = nm]$$

$$\therefore \qquad k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \dots\dots\dots . + r_n^2}{n}}$$

Hence radius of gyration of a body about a given axis is equal to root mean square distance of the constituent particles of the body from the given axis.

(1) Radius of gyration (k) depends on shape and size of the body, position and configuration of the axis of rotation, distribution of mass of the body *w.r.t.* the axis of rotation.

(2) Radius of gyration (k) does not depends on the mass of body.

(3) Dimension $[M^0 L^1 T^0]$.

(4) S.I. unit : *Meter*.

(5) Significance of radius of gyration : Through this concept a real body (particularly irregular) is replaced by a point mass for dealing its rotational motion.

Example : In case of a disc rotating about an axis through its centre of mass and perpendicular to its plane

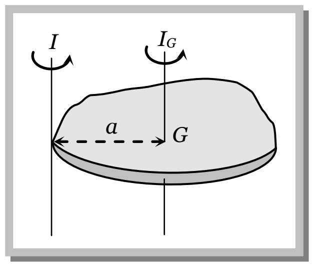
$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{(1/2)MR^2}{M}} = \frac{R}{\sqrt{2}}$$

So instead of disc we can assume a point mass M at a distance $(R/\sqrt{2})$ from the axis of rotation for dealing the rotational motion of the disc.

Note : ❑ For a given body inertia is constant whereas moment of inertia is variable.

7.9 Theorem of Parallel Axes

Moment of inertia of a body about a given axis I is equal to the sum of moment of inertia of the body about an axis parallel to given axis and passing through centre of mass of the body I_g and Ma^2 where M is the mass of the body and a is the perpendicular distance between the two axes.



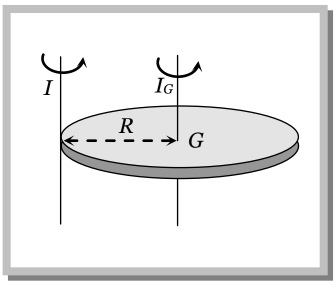
$$I = I_g + Ma^2$$

Example: Moment of inertia of a disc about an axis through its centre and perpendicular to the plane is $\frac{1}{2}MR^2$, so moment of inertia about an axis through its tangent and perpendicular to the plane will be

$$I = I_g + Ma^2$$

$$I = \frac{1}{2}MR^2 + MR^2$$

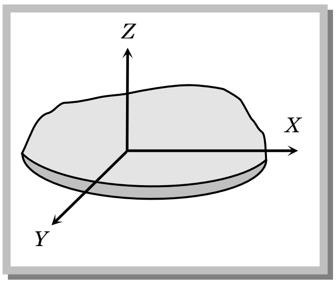
$$\therefore \qquad I = \frac{3}{2}MR^2$$



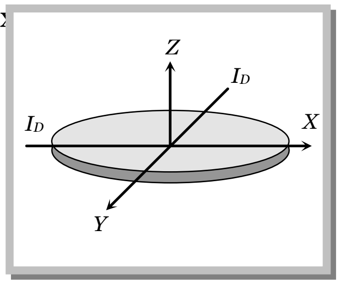
7.10 Theorem of Perpendicular Axes

According to this theorem the sum of moment of inertia of a plane lamina about two mutually perpendicular axes lying in its plane is equal to its moment of inertia about an axis perpendicular to the plane of lamina and passing through the point of intersection of first two axes.

$$I_z = I_x + I_y$$



Example : Moment of inertia of a disc about an axis through its centre of mass and perpendicular to its plane is $\frac{1}{2}MR^2$, so if the disc is in *x–y* plane then by theorem of perpendicular axes



i.e. $I_z = I_x + I_y$

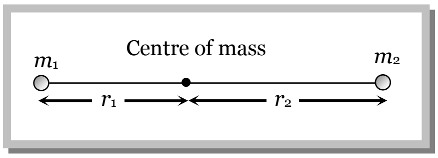
$\Rightarrow$ $\frac{1}{2}MR^2 = 2I_D$ [As ring is symmetrical body so $I_x = I_y = I_D$]

$\Rightarrow$ $I_D = \frac{1}{4}MR^2$

Note : ❑ In case of symmetrical two-dimensional bodies as moment of inertia for all axes passing through the centre of mass and in the plane of body will be same so the two axes in the plane of body need not be perpendicular to each other.

7.11 Moment of Inertia of Two Point Masses About Their Centre of Mass

Let m_1 and m_2 be two masses distant r from each-other and r_1 and r_2 be the distances of their centre of mass from m_1 and m_2 respectively, then



(1) $r_1 + r_2 = r$

(2) $m_1 r_1 = m_2 r_2$

(3) $r_1 = \frac{m_2}{m_1 + m_2} r$ and $r_2 = \frac{m_1}{m_1 + m_2} r$

(4) $I = m_1 r_1^2 + m_2 r_2^2$

(5) $I = \left[\frac{m_1 m_2}{m_1 + m_2}\right] r^2 = \mu r^2$ [where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is known as reduced mass $\mu < m_1$ and $\mu < m_2$.]

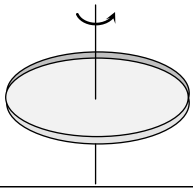
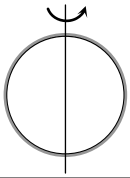
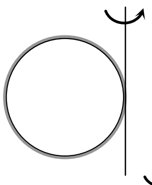
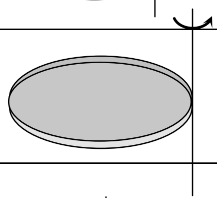
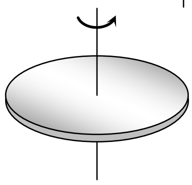
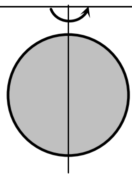
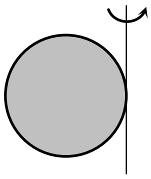
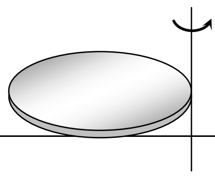
(6) In diatomic molecules like H_2, HCl *etc.* moment of inertia about their centre of mass is derived from above formula.

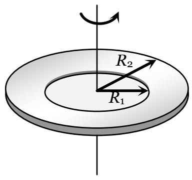
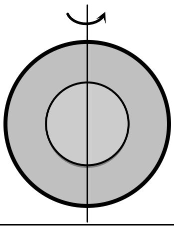
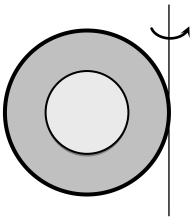
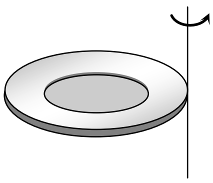
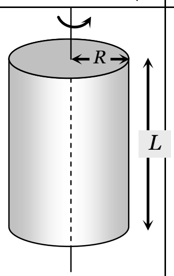
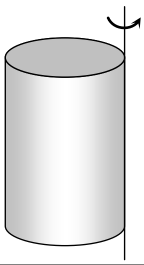
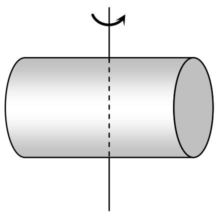
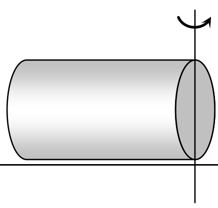
7.12 Analogy Between Tranlatory Motion and Rotational Motion

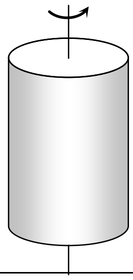
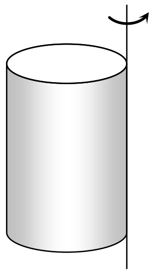
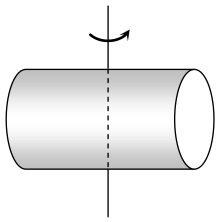
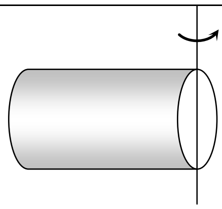
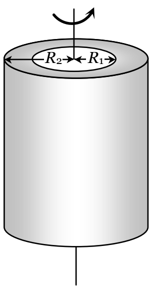
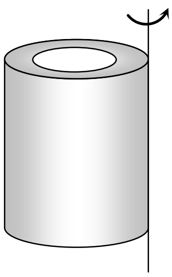
Translatory motion		Rotatory motion	
Mass	(m)	Moment of Inertia	(I)
Linear momentum	$P = mv$ $P = \sqrt{2mE}$	Angular Momentum	$L = I\omega$ $L = \sqrt{2IE}$
Force	$F = ma$	Torque	$\tau = I\alpha$

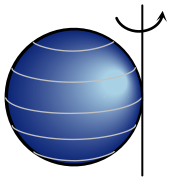
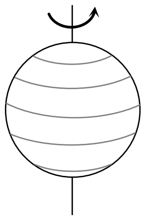
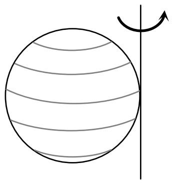
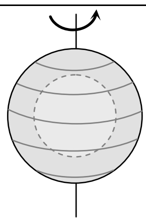
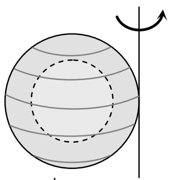
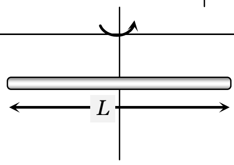
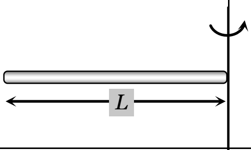
Kinetic energy	$E = \frac{1}{2}mv^2$ $E = \frac{P^2}{2m}$	$E = \frac{1}{2}I\omega^2$ $E = \frac{L^2}{2I}$

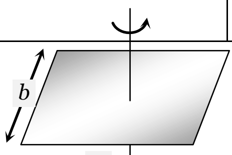
7.13 Moment of Inertia of Some Standard Bodies About Different Axes

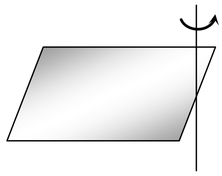
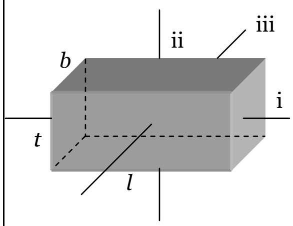
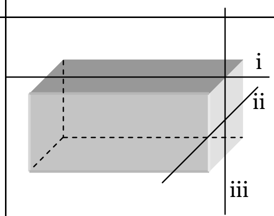
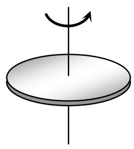
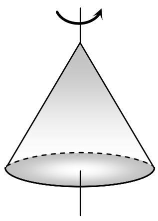
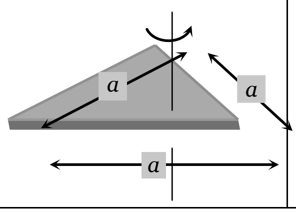
Body	Axis of Rotation	Figure	Moment of inertia	*k*	k^2/R^2
Ring	About an axis passing through C.G. and perpendicular to its plane		MR^2	R	1
Ring	About its diameter		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Ring	About a tangential axis in its own plane		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$	$\frac{3}{2}$
Ring	About a tangential axis perpendicular to its own plane		$2MR^2$	$\sqrt{2}R$	2
Disc	About an axis passing through C.G. and perpendicular to its plane		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Disc	About its Diameter		$\frac{1}{4}MR^2$	$\frac{R}{2}$	$\frac{1}{4}$
Disc	About a tangential axis in its own plane		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$	$\frac{5}{4}$
Disc	About a tangential axis perpendicular to		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$	$\frac{3}{2}$

	its own plane				
Annular disc inner radius = R_1 and outer radius = R_2	Passing through the centre and perpendicular to the plane		$\frac{M}{2}[R_1^2 + R_2^2]$	–	–
Annular disc	Diameter		$\frac{M}{4}[R_1^2 + R_2^2]$	–	–
Annular disc	Tangential and Parallel to the diameter		$\frac{M}{4}[5R_1^2 + R_2^2]$	–	–
Annular disc	Tangential and perpendicular to the plane		$\frac{M}{2}[3R_1^2 + R_2^2]$	–	–
Solid cylinder	About its own axis		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Solid cylinder	Tangential (Generator)		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$	$\frac{3}{2}$
Solid cylinder	About an axis passing through its C.G. and perpendicular to its own axis		$M\left[\frac{L^2}{12} + \frac{R^2}{4}\right]$	$\sqrt{\frac{L^2}{12} + \frac{R^2}{4}}$	
Solid cylinder	About the diameter of one of faces of the cylinder		$M\left[\frac{L^2}{3} + \frac{R^2}{4}\right]$	$\sqrt{\frac{L^2}{3} + \frac{R^2}{4}}$	

Cylindrical shell	About its own axis		MR^2	R	1
Cylindrical shell	Tangential (Generator)		$2MR^2$	$\sqrt{2}R$	2
Cylindrical shell	About an axis passing through its C.G. and perpendicular to its own axis		$M\left[\frac{L^2}{12}+\frac{R^2}{2}\right]$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{2}}$	
Cylindrical shell	About the diameter of one of faces of the cylinder		$M\left[\frac{L^2}{3}+\frac{R^2}{2}\right]$	$\sqrt{\frac{L^2}{3}+\frac{R^2}{2}}$	
Hollow cylinder with inner radius = R_1 and outer radius = R_2	Axis of cylinder		$\frac{M}{2}(R_1^2+R_2^2)$		
Hollow cylinder with inner radius = R_1 and outer radius = R_2	Tangential		$\frac{M}{2}(R_1^2+3R_2^2)$		

Solid Sphere	About its diametric axis		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$	$\frac{2}{5}$
Solid sphere	About a tangential axis		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$	$\frac{7}{5}$
Spherical shell	About its diametric axis		$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$	$\frac{2}{3}$
Spherical shell	About a tangential axis		$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$	$\frac{5}{3}$
Hollow sphere of inner radius R_1 and outer radius R_2	About its diametric axis		$\frac{2}{5}M\left[\frac{R_2^5 - R_1^5}{R_2^3 - R_1^3}\right]$		
Hollow sphere	Tangential		$\frac{2M[R_2^5 - R_1^5]}{5(R_2^3 - R_1^3)} + MR_2^2$		
Long thin rod	About on axis passing through its centre of mass and perpendicular to the rod.		$\frac{ML^2}{12}$	$\frac{L}{\sqrt{12}}$	
Long thin rod	About an axis passing through its edge and perpendicular to the rod		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$	
Rectangular lamina of length l and	Passing through the centre of mass		$\frac{M}{12}[l^2 + b^2]$		



breadth b	and perpendicular to the plane				
Rectangular lamina	Tangential perpendicular to the plane and at the mid-point of breadth		$\frac{M}{12}[4l^2 + b^2]$		
Rectangular lamina	Tangential perpendicular to the plane and at the mid-point of length		$\frac{M}{12}[l^2 + 4b^2]$		
Rectangular parallelopiped length l, breadth b, thickness t	Passing through centre of mass and parallel to (i) Length (x) (ii) breadth (z) (iii) thickness (y)		(i) $\frac{M[b^2 + t^2]}{12}$ (ii) $\frac{M[l^2 + t^2]}{12}$ (iii) $\frac{M[b^2 + l^2]}{12}$		
Rectangular parallelepiped length l, breath b, thickness t	Tangential and parallel to (i) length (x) (ii) breadth (y) (iii) thickness(z)		(i) $\frac{M}{12}[3l^2 + b^2 + t^2]$ (ii) $\frac{M}{12}[l^2 + 3b^2 + t^2]$ (iii) $\frac{M}{12}[l^2 + b^2 + 3t^2]$		
Elliptical disc of semimajor axis = a and semiminor axis = b	Passing through CM and perpendicular to the plane		$\frac{M}{4}[a^2 + b^2]$		
Solid cone of radius R and height h	Axis joining the vertex and centre of the base		$\frac{3}{10}MR^2$		
Equilateral triangular lamina with side a	Passing through CM and perpendicular to the plane		$\frac{Ma^2}{6}$		

Right angled triangular lamina of sides a, b, c	Along the edges		(1) $\frac{Mb^2}{6}$ (2) $\frac{Ma^2}{6}$ (3) $\frac{M}{6}\left[\frac{a^2b^2}{a^2+b^2}\right]$		

Sample problem based on moment of inertia

Problem 14. Five particles of mass = 2 *kg* are attached to the rim of a circular disc of radius 0.1 *m* and negligible mass. Moment of inertia of the system about the axis passing through the centre of the disc and perpendicular to its plane is

(a) 1 $kg\ m^2$ (b) 0.1 $kg\ m^2$ (c) 2 $kg\ m^2$ (d) 0.2 $kg\ m^2$

Solution: (b) We will not consider the moment of inertia of disc because it doesn't have any mass so moment of inertia of five particle system $I = 5\,mr^2 = 5 \times 2 \times (0.1)^2 = 0.1\,kg\text{-}m^2$.

Problem 15. A circular disc X of radius R is made from an iron plate of thickness t, and another disc Y of radius $4R$ is made from an iron plate of thickness $\frac{t}{4}$. Then the relation between the moment of inertia I_X and I_Y is **[AIEEE 2003]**

(a) $I_Y = 64 I_X$ (b) $I_Y = 32 I_X$ (c) $I_Y = 16 I_X$ (d) $I_Y = I_X$

Solution: (a) Moment of Inertia of disc $I = \frac{1}{2}MR^2 = \frac{1}{2}(\pi R^2 t\rho)R^2 = \frac{1}{2}\pi t\rho R^4$

[As $M = V \times \rho = \pi R^2 t\rho$ where $t =$ thickness, $\rho =$ density]

$\therefore \quad \frac{I_y}{I_x} = \frac{t_y}{t_x}\left(\frac{R_y}{R_x}\right)^4$ [If ρ = constant]

$\Rightarrow \quad \frac{I_y}{I_x} = \frac{1}{4}(4)^4 = 64$ [Given $R_y = 4R_x$, $t_y = \frac{t_x}{4}$]

$\Rightarrow \quad I_y = 64 I_x$

Problem 16. Moment of inertia of a uniform circular disc about a diameter is I. Its moment of inertia about an axis perpendicular to its plane and passing through a point on its rim will be **[UPSEAT 2002]**

(a) $5\,I$ (b) $6\,I$ (c) $3\,I$ (d) $4\,I$

Solution: (b) Moment of inertia of disc about a diameter = $\frac{1}{4}MR^2 = I$ (given) $\therefore\ MR^2 = 4I$

Now moment of inertia of disc about an axis perpendicular to its plane and passing through a point on its rim

$$= \frac{3}{2}MR^2 = \frac{3}{2}(4I) = 6I.$$

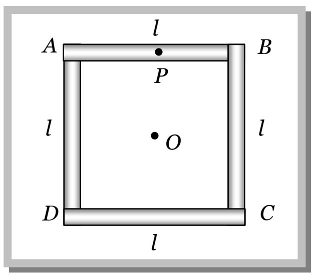
Problem 17. Four thin rods of same mass M and same length l, form a square as shown in figure. Moment of inertia of this system about an axis through centre O and perpendicular to its plane is

(a) $\frac{4}{3}Ml^2$

(b) $\frac{Ml^2}{3}$

(c) $\frac{Ml^2}{6}$

(d) $\frac{2}{3}Ml^2$

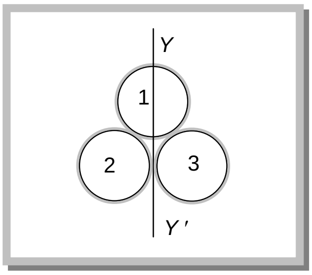


Solution: (a) Moment of inertia of rod AB about point $P = \frac{1}{12}Ml^2$

M.I. of rod AB about point $O = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{1}{3}Ml^2$ [by the theorem of parallel axis]

and the system consists of 4 rods of similar type so by the symmetry $I_{System} = \frac{4}{3}Ml^2$.

***Problem* 18.** Three rings each of mass M and radius R are arranged as shown in the figure. The moment of inertia of the system about YY' will be



(a) $3\,MR^2$

(b) $\frac{3}{2}MR^2$

(c) $5\,MR^2$

(d) $\frac{7}{2}MR^2$

Solution: (d) M.I of system about YY' $I = I_1 + I_2 + I_3$

where I_1 = moment of inertia of ring about diameter, $I_2 = I_3$ = M.I. of inertia of ring about a tangent in a plane

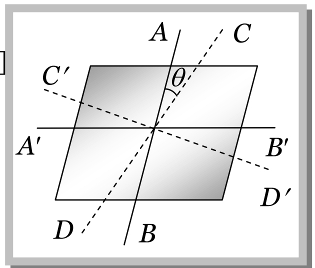
$\therefore \quad I = \frac{1}{2}mR^2 + \frac{3}{2}mR^2 + \frac{3}{2}mR^2 = \frac{7}{2}mR^2$

***Problem* 19.** Let l be the moment of inertia of an uniform square plate about an axis AB that passes through its centre and is parallel to two of its sides. CD is a line in the plane of the plate that passes through the centre of the plate and makes an angle θ with AB. The moment of inertia of the plate about the axis CD is then equal to

[IIT-JEE 1998]

(a) l (b) $l\sin^2\theta$ (c) $l\cos^2\theta$ (d) $l\cos^2\frac{\theta}{2}$

Solution: (a) Let I_Z is the moment of inertia of square plate about the axis which is passing through the centre and perpendicular to the plane.



$I_Z = I_{AB} + I_{A'B'} = I_{CD} + I_{C'D'}$ [By the theorem of perpendicular axis]

$I_Z = 2I_{AB} = 2I_{A'B'} = 2I_{CD} = 2I_{C'D'}$

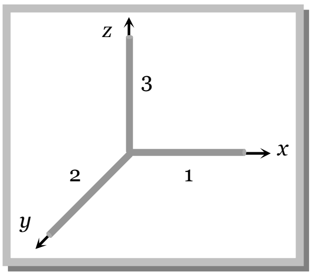
[As AB, $A'B'$ and CD, $C'D'$ are symmetric axis]

Hence $I_{CD} = I_{AB} = l$

***Problem* 20.** Three rods each of length L and mass M are placed along X, Y and Z-axes in such a way that one end of each of the rod is at the origin. The moment of inertia of this system about Z axis is

(a) $\frac{2ML^2}{3}$ (b) $\frac{4ML^2}{3}$ (c) $\frac{5ML^2}{3}$ (d) $\frac{ML^2}{3}$

Solution: (a) Moment of inertia of the system about z-axis can be find out by calculating the moment of inertia of individual rod about z-axis



$I_1 = I_2 = \frac{ML^2}{3}$ because z-axis is the edge of rod 1 and 2

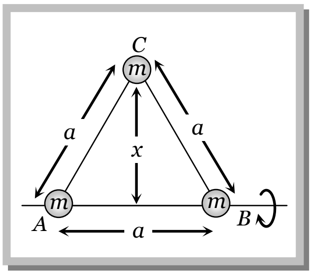
and $I_3 = 0$ because rod in lying on z-axis

$\therefore \quad I_{system} = I_1 + I_2 + I_3 = \frac{ML^2}{3} + \frac{ML^2}{3} + 0 = \frac{2ML^2}{3}$.

Problem 21. Three point masses each of mass m are placed at the corners of an equilateral triangle of side a. Then the moment of inertia of this system about an axis passing along one side of the triangle is **[AIIMS 1995]**

(a) ma^2 (b) $3ma^2$ (c) $\frac{3}{4}ma^2$ (d) $\frac{2}{3}ma^2$

Solution: (c) The moment of inertia of system about AB side of triangle

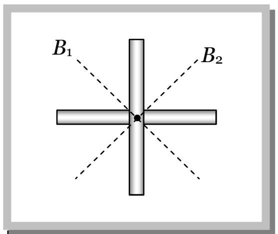


$I = I_A + I_B + I_C$

$= 0 + 0 + mx^2$

$= m\left(\frac{a\sqrt{3}}{2}\right)^2 = \frac{3}{4}ma^2$

Problem 22. Two identical rods each of mass M. and length l are joined in crossed position as shown in figure. The moment of inertia of this system about a bisector would be



(a) $\frac{Ml^2}{6}$ (b) $\frac{Ml^2}{12}$

(c) $\frac{Ml^2}{3}$ (d) $\frac{Ml^2}{4}$

Solution: (b) Moment of inertia of system about an axes which is perpendicular to plane of rods and passing through the common centre of rods $I_z = \frac{Ml^2}{12} + \frac{Ml^2}{12} = \frac{Ml^2}{6}$

Again from perpendicular axes theorem $I_z = I_{B_1} + I_{B_2} = 2I_{B_1} = 2I_{B_2} = \frac{Ml^2}{6}$ [As $I_{B_1} = I_{B_2}$]

$\therefore \quad I_{B_1} = I_{B_2} = \frac{Ml^2}{12}$.

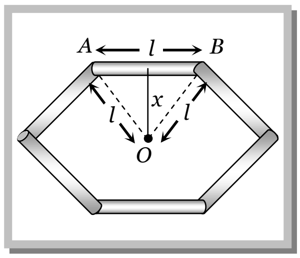
Problem 23. The moment of inertia of a rod of length l about an axis passing through its centre of mass and perpendicular to rod is I. The moment of inertia of hexagonal shape formed by six such rods, about an axis passing through its centre of mass and perpendicular to its plane will be

(a) $16I$ (b) $40\,I$ (c) $60\,I$ (d) $80\,I$

Solution: (c) Moment of inertia of rod AB about its centre and perpendicular to the length $= \frac{ml^2}{12} = I$ $\therefore$

$ml^2 = 12I$

Now moment of inertia of the rod about the axis which is passing through O and perpendicular to the plane of hexagon $I_{rod} = \frac{ml^2}{12} + mx^2$ [From the theorem of parallel axes]



$= \frac{ml^2}{12} + m\left(\frac{\sqrt{3}}{2}l\right)^2 = \frac{5ml^2}{6}$

Now the moment of inertia of system $I_{system} = 6 \times I_{rod} = 6 \times \frac{5ml^2}{6} = 5ml^2$

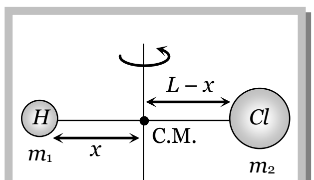
$I_{system} = 5\,(12\,I) = 60\,I$ [As $ml^2 = 12I$]

Problem 24. The moment of inertia of HCl molecule about an axis passing through its centre of mass and perpendicular to the line joining the H^+ and Cl^- ions will be, if the interatomic distance is 1 Å

(a) $0.61 \times 10^{-47}\,kg.m^2$ (b) $1.61 \times 10^{-47}\,kg.m^2$ (c) $0.061 \times 10^{-47}\,kg.m^2$ (d) 0

Solution: (b) If r_1 and r_2 are the respective distances of particles m_1 and m_2 from the centre of mass then

$m_1 r_1 = m_2 r_2 \Rightarrow 1 \times x = 35.5 \times (L - x) \Rightarrow x = 35.5(1 - x)$



$\Rightarrow x = 0.973 Å$ and $L - x = 0.027 Å$

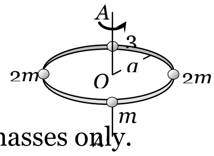
Moment of inertia of the system about centre of mass $I = m_1 x^2 + m_2 (L - x)^2$

$I = 1 amu \times (0.973 Å)^2 + 35.5\, amu \times (0.027 Å)^2$

Substituting 1 *a.m.u.* = 1.67×10^{-27} *kg* and 1 Å = 10^{-10} *m*

$I = 1.62 \times 10^{-47} kg\ m^2$

***Problem* 25.** Four masses are joined to a light circular frame as shown in the figure. The radius of gyration of this system about an axis passing through the centre of the circular frame and perpendicular to its plane would be



(a) $a/\sqrt{2}$ (b) $a/2$

(c) a (d) $2a$

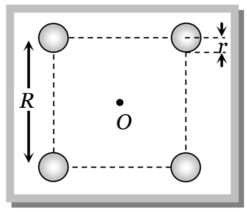
Solution: (c) Since the circular frame is massless so we will consider moment of inertia of four masses only.

$I = ma^2 + 2ma^2 + 3ma^2 + 2ma^2 = 8ma^2$ .....(i)

Now from the definition of radius of gyration $I = 8mk^2$ .....(ii)

comparing (i) and (ii) radius of gyration $k = a$.

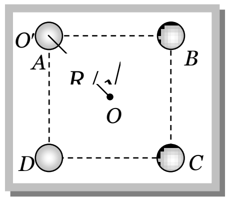
***Problem* 26.** Four spheres, each of mass M and radius r are situated at the four corners of square of side R. The moment of inertia of the system about an axis perpendicular to the plane of square and passing through its centre will be



(a) $\frac{5}{2} M(4r^2 + 5R^2)$ (b) $\frac{2}{5} M(4r^2 + 5R^2)$

(c) $\frac{2}{5} M(4r^2 + 5r^2)$ (d) $\frac{5}{2} M(4r^2 + 5r^2)$

Solution: (b) *M. I.* of sphere *A* about its diameter $I_{O'} = \frac{2}{5} Mr^2$



Now *M.I.* of sphere *A* about an axis perpendicular to the plane of square and passing through its centre will be

$I_O = I_{O'} + M\left(\frac{R}{\sqrt{2}}\right)^2 = \frac{2}{5} Mr^2 + \frac{MR^2}{2}$ [by the theorem of parallel axis]

Moment of inertia of system (i.e. four sphere)= $4 I_O = 4\left[\frac{2}{5} Mr^2 + \frac{MR^2}{2}\right] = \frac{2}{5} M\left[4r^2 + 5R^2\right]$

***Problem* 27.** The moment of inertia of a solid sphere of density ρ and radius *R* about its diameter is

(a) $\frac{105}{176} R^5 \rho$ (b) $\frac{105}{176} R^2 \rho$ (c) $\frac{176}{105} R^5 \rho$ (d) $\frac{176}{105} R^2 \rho$

Solution: (c) Moment of inertia of sphere about it diameter $I = \frac{2}{5} MR^2 = \frac{2}{5}\left(\frac{4}{3}\pi R^3 \rho\right) R^2$ [As $M = V\rho = \frac{4}{3}\pi R^3 \rho$]

$I = \frac{8\pi}{15} R^5 \rho = \frac{8 \times 22}{15 \times 7} R^5 \rho = \frac{176}{105} R^5 \rho$

***Problem* 28.** Two circular discs A and B are of equal masses and thickness but made of metals with densities d_A and d_B $(d_A > d_B)$. If their moments of inertia about an axis passing through centres and normal to the circular faces be I_A and I_B, then

(a) $I_A = I_B$ (b) $I_A > I_B$ (c) $I_A < I_B$ (d) $I_A >=< I_B$

Solution : (c) Moment of inertia of circular disc about an axis passing through centre and normal to the circular face

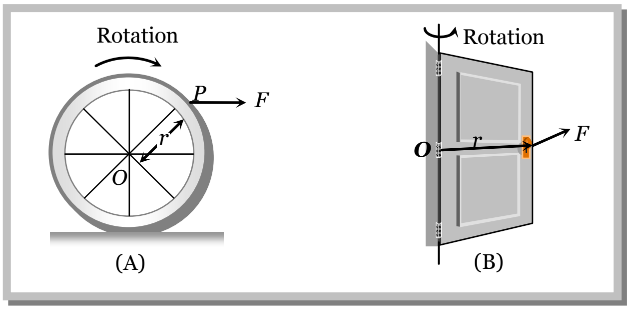
$$I = \frac{1}{2}MR^2 = \frac{1}{2}M\left(\frac{M}{\pi t\rho}\right) \quad \left[\text{As } M = V\rho = \pi R^2 t\rho \ \therefore\ R^2 = \frac{M}{\pi t \rho}\right]$$

$$\Rightarrow \quad I = \frac{M^2}{2\pi t\rho} \quad \text{or} \quad I \propto \frac{1}{\rho} \text{ If mass and thickness are constant.}$$

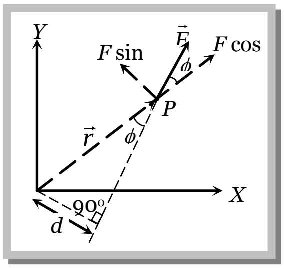
So, in the problem $\frac{I_A}{I_B} = \frac{d_B}{d_A}$ $\therefore I_A < I_B$ [As $d_A > d_B$]

7.14 Torque

If a pivoted, hinged or suspended body tends to rotate under the action of a force, it is said to be acted upon by a torque. or The turning effect of a force about the axis of rotation is called moment of force or torque due to the force.



If the particle rotating in xy plane about the origin under the effect of force $\vec{F}$ and at any instant the position vector of the particle is $\vec{r}$ then,



Torque $\vec{\tau} = \vec{r} \times \vec{F}$

$\tau = rF\sin\phi$

[where ϕ is the angle between the direction of $\vec{r}$ and $\vec{F}$]

(1) Torque is an axial vector. *i.e.,* its direction is always perpendicular to the plane containing vector $\vec{r}$ and $\vec{F}$ in accordance with right hand screw rule. For a given figure the sense of rotation is anti-clockwise so the direction of torque is perpendicular to the plane, outward through the axis of rotation.

(2) Rectangular components of force

$\vec{F}_r = F\cos\phi$ = radial component of force , $\vec{F}_\phi = F\sin\phi$ = transverse component of force

As $\tau = rF\sin\phi$

or $\tau = rF_\phi$ = (position vector) × (transverse component of force)

Thus the magnitude of torque is given by the product of transverse component of force and its perpendicular distance from the axis of rotation *i.e.*, Torque is due to transverse component of force only.

(3) As $\tau = rF\sin\phi$

or $\tau = F(r\sin\phi) = Fd$ [As $d = r\sin\phi$ from the figure]

i.e. Torque = Force × Perpendicular distance of line of action of force from the axis of rotation.

Torque is also called as moment of force and d is called moment or lever arm.

(4) Maximum and minimum torque : As $\vec{\tau} = \vec{r} \times \vec{F}$ or $\tau = rF\sin\phi$

$\tau_{maximum} = rF$	When $\lvert \sin\phi \rvert = \max = 1$ *i.e.,* $\phi = 90°$	$\vec{F}$ is perpendicular to $\vec{r}$
$\tau_{minimum} = 0$	When $\lvert \sin\phi \rvert = \min = 0$ *i.e.* $\phi = 0°$ or $180°$	$\vec{F}$ is collinear to $\vec{r}$

(5) For a given force and angle, magnitude of torque depends on r. The more is the value of r, the more will be the torque and easier to rotate the body.

Example : (i) Handles are provided near the free edge of the Planck of the door.

(ii) The handle of screw driver is taken thick.

(iii) In villages handle of flourmill is placed near the circumference.

(iv) The handle of hand-pump is kept long.

(v) The arm of wrench used for opening the tap, is kept long.

(6) Unit : *Newton-metre* (M.K.S.) and *Dyne-cm* (C.G.S.)

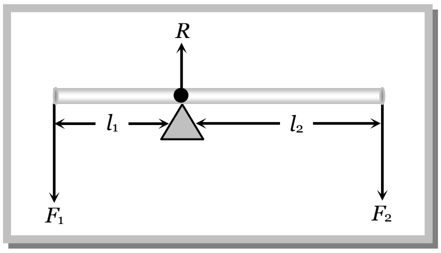
(7) Dimension : $[ML^2T^{-2}]$.

(8) If a body is acted upon by more than one force, the total torque is the vector sum of each torque.

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + ...$$

(9) A body is said to be in rotational equilibrium if resultant torque acting on it is zero *i.e.* $\Sigma \vec{\tau} = 0$.

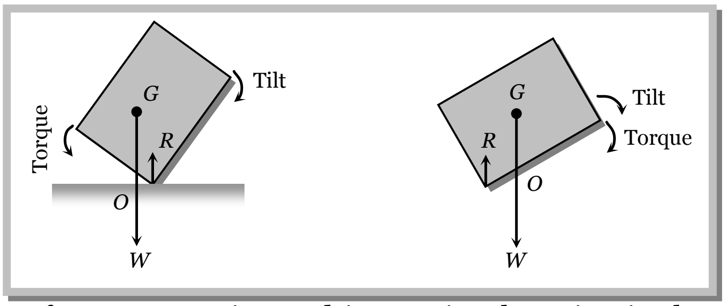
(10) In case of beam balance or see-saw the system will be in rotational equilibrium if,



$\vec{\tau}_1 + \vec{\tau}_2 = 0$ or $F_1 l_1 - F_2 l_2 = 0$ $\therefore$ $F_1 l_1 = F_2 l_2$

However if, $\vec{\tau}_1 > \vec{\tau}_2$, L.H.S. will move downwards and if $\vec{\tau}_1 < \vec{\tau}_2$. R.H.S. will move downward. and the system will not be in rotational equilibrium.

(11) On tilting, a body will restore its initial position due to torque of weight about the point O till the line of action of weight passes through its base on tilting, a body will topple due to torque of weight about O, if the line of action of weight does not pass through the base.



(12) Torque is the cause of rotatory motion and in rotational motion it plays same role as force plays in translatory motion *i.e.,* torque is rotational analogue of force. This all is evident from the following correspondences between rotatory and translatory motion.

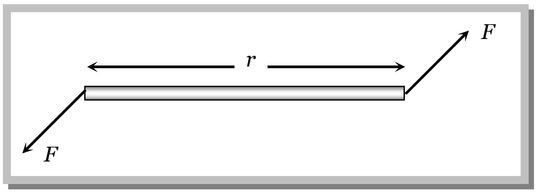
Rotatory Motion	**Translatory Motion**
$\vec{\tau} = I\vec{\alpha}$	$\vec{F} = m\vec{a}$
$W = \int \vec{\tau} \cdot d\vec{\theta}$	$W = \int \vec{F} \cdot d\vec{s}$

$P = \vec{\tau} \cdot \vec{\omega}$	$P = \vec{F} \cdot \vec{v}$
$\vec{\tau} = \frac{d\vec{L}}{dt}$	$\vec{F} = \frac{d\vec{P}}{dt}$

7.15 Couple

A special combination of forces even when the entire body is free to move can rotate it. This combination of forces is called a couple.

(1) A couple is defined as combination of two equal but oppositely directed force not acting along the same line. The effect of couple is known by its moment of couple or torque by a couple $\vec{\tau} = \vec{r} \times \vec{F}$.

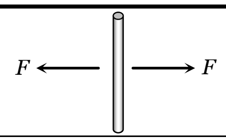
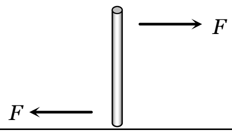
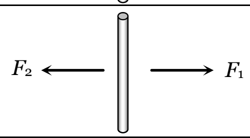
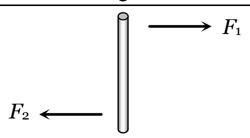


(2) Generally both couple and torque carry equal meaning. The basic difference between torque and couple is the fact that in case of couple both the forces are externally applied while in case of torque one force is externally applied and the other is reactionary.

(3) Work done by torque in twisting the wire $W = \frac{1}{2} C\theta^2$.

Where $\tau = C\theta$; C is known as twisting coefficient or couple per unit twist.

7.16 Translatory and Rotatory Equilibrium

Forces are equal and act along the same line.		$\sum F = 0$ and $\sum \tau = 0$	Body will remain stationary if initially it was at rest.
Forces are equal and does not act along the same line.		$\sum F = 0$ and $\sum \tau \neq 0$	Rotation *i.e.* spinning.
Forces are unequal and act along the same line.		$\sum F \neq 0$ and $\sum \tau = 0$	Translation *i.e.* slipping or skidding.
Forces are unequal and does not act along the same line.		$\sum F \neq 0$ and $\sum \tau \neq 0$	Rotation and translation both *i.e.* rolling.

Sample problems based on torque and couple

***Problem* 29.** A force of $(2\hat{i} - 4\hat{j} + 2\hat{k})N$ acts at a point $(3\hat{i} + 2\hat{j} - 4\hat{k})$ *metre* from the origin. The magnitude of torque is

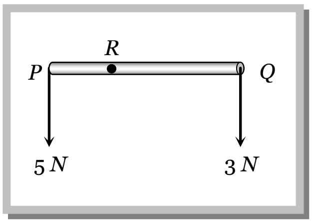
(a) Zero (b) 24.4 *N-m* (c) 0.244 *N-m* (d) 2.444 *N-m*

Solution: (b) $\vec{F} = (2\hat{i} - 4\hat{j} + 2\hat{k})N$ and $\vec{r} = (3i + 2 - 4\hat{k})$ *meter*

Torque $\vec{\tau}=\vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -4 \\ 2 & -4 & 2 \end{vmatrix} \Rightarrow \vec{\tau}=-12\hat{i}-14\hat{j}-16\hat{k}$ and $|\vec{\tau}|=\sqrt{(-12)^2+(-14)^2+(-16)^2}=$

24.4 *N-m*

***Problem* 30.** The resultant of the system in the figure is a force of $8N$ parallel to the given force through R. The value of PR equals to



(a) 1/4 *RQ*

(b) 3/8 *RQ*

(c) 3/5 *RQ*

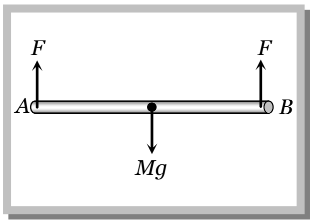
(d) 2/5 *RQ*

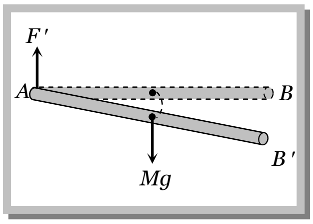
Solution: (c) By taking moment of forces about point *R*, $5\times PR-3\times RQ=0 \Rightarrow PR=\frac{3}{5}RQ$.

***Problem* 31.** A horizontal heavy uniform bar of weight *W* is supported at its ends by two men. At the instant, one of the men lets go off his end of the rod, the other feels the force on his hand changed to

(a) W (b) $\frac{W}{2}$ (c) $\frac{3W}{4}$ (d) $\frac{W}{4}$

Solution: (d) Let the mass of the rod is M $\therefore$ Weight $(W) = Mg$





Initially for the equilibrium $F+F=Mg \Rightarrow F=Mg/2$

When one man withdraws, the torque on the rod

$$\tau = I\alpha = Mg\frac{l}{2}$$

$$\Rightarrow \frac{Ml^2}{3}\alpha = Mg\frac{l}{2} \qquad [\text{As } I = Ml^2/3]$$

$\Rightarrow$ Angular acceleration $\alpha=\frac{3}{2}\frac{g}{l}$

and linear acceleration $a=\frac{l}{2}\alpha=\frac{3g}{4}$

Now if the new normal force at *A* is F' then $Mg-F'=Ma$

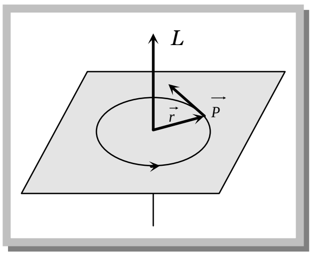
$$\Rightarrow F'=Mg-Ma=Mg-\frac{3Mg}{4}=\frac{Mg}{4}=\frac{W}{4}.$$

7.17 Angular Momentum

The turning momentum of particle about the axis of rotation is called the angular momentum of the particle.

or

The moment of linear momentum of a body with respect to any axis of rotation is known as angular momentum. If $\vec{P}$ is the linear momentum of particle and $\vec{r}$ its position vector from the point of rotation then angular momentum.



$$\vec{L}=\vec{r}\times\vec{P}$$

$$\vec{L}=rP\sin\phi\,\hat{n}$$

Angular momentum is an axial vector *i.e.* always directed perpendicular to the plane of rotation and along the axis of rotation.

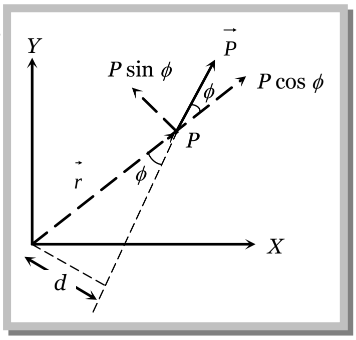
(1) S.I. Unit : *kg-m²-s⁻¹* or *J-sec*.

(2) Dimension : $[ML^2T^{-1}]$ and it is similar to Planck's constant (h).

(3) In cartesian co-ordinates if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$

Then $\vec{L} = \vec{r} \times \vec{P} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix} = (yP_z - zP_y)\hat{i} - (xP_z - zP_x)\hat{j} + (xP_y - yP_x)\hat{k}$

(4) As it is clear from the figure radial component of momentum $\vec{P}_r = P\cos$



Transverse component of momentum $\vec{P}_\phi = P\sin\phi$

So magnitude of angular momentum $L = r\,P\sin\phi$

$$L = r\,P_\phi$$

$\therefore$ Angular momentum = Position vector × Transverse component of angular momentum

i.e., The radial component of linear momentum has no role to play in angular momentum.

(5) Magnitude of angular momentum $L = P\ \ (r\sin\phi) = L = Pd$ [As $d = r\sin\phi$ from the figure.]

$\therefore$ Angular momentum = (Linear momentum) × (Perpendicular distance of line of action of force from the axis of rotation)

(6) Maximum and minimum angular momentum : We know $\vec{L} = \vec{r} \times \vec{P}$

$\therefore$ $\vec{L} = m[\vec{r} \times \vec{v}] = m\,v\,r\sin\phi = P\,r\sin\phi$ [As $\vec{P} = m\,\vec{v}$]

$L_{maximum} = mvr$	When $\lvert\sin\phi\rvert = \max = 1$ *i.e.*, $\phi = 90°$	$\vec{v}$ is perpendicular to $\vec{r}$
$L_{minimum} = 0$	When $\lvert\sin\phi\rvert = \min = 0$ *i.e.* $\phi = 0°$ or $180°$	$\vec{v}$ is parallel or anti-parallel to $\vec{r}$

(7) A particle in translatory motion always have an angular momentum unless it is a point on the line of motion because $L = mvr\sin\phi$ and $L > 1$ if $\phi \neq 0^o$ or 180^o

(8) In case of circular motion, $\vec{L} = \vec{r} \times \vec{P} = m(\vec{r} \times \vec{v}) = mvr\sin\phi$

$\therefore$ $L = mvr = mr^2\omega$ [As $\vec{r} \perp \vec{v}$ and $v = r\omega$]

or $L = I\omega$ [As $mr^2 = I$]

In vector form $\vec{L} = I\vec{\omega}$

(9) From $\vec{L} = I\vec{\omega}$ $\therefore$ $\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha} = \vec{\tau}$ [As $\frac{d\vec{\omega}}{dt} = \vec{\alpha}$ and $\vec{\tau} = I\vec{\alpha}$]

i.e. the rate of change of angular momentum is equal to the net torque acting on the particle. [Rotational analogue of Newton's second law]

(10) If a large torque acts on a particle for a small time then 'angular impulse' of torque is given by

$$\vec{J} = \int \vec{\tau}\, dt = \vec{\tau}_{av} \int_{t_1}^{t_2} dt$$

or Angular impulse $\vec{J} = \vec{\tau}_{av}\,\Delta t = \Delta \vec{L}$

$\therefore$ Angular impulse = Change in angular momentum

(11) The angular momentum of a system of particles is equal to the vector sum of angular momentum of each particle *i.e.*, $\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + ... + \vec{L}_n$.

(12) According to Bohr theory angular momentum of an electron in n^{th} orbit of atom can be taken as,

$$L = n\frac{h}{2\pi}$$ [where n is an integer used for number of orbit]

7.18 Law of Conservation of Angular Momentum

Newton's second law for rotational motion $\vec{\tau} = \frac{d\vec{L}}{dt}$

So if the net external torque on a particle (or system) is zero then $\frac{d\vec{L}}{dt} = 0$

i.e. $\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + ...$ = constant.

Angular momentum of a system (may be particle or body) remains constant if resultant torque acting on it zero.

As $L = I\omega$ so if $\vec{\tau} = 0$ then $I\omega = \text{constant}$ $\therefore$ $I \propto \frac{1}{\omega}$

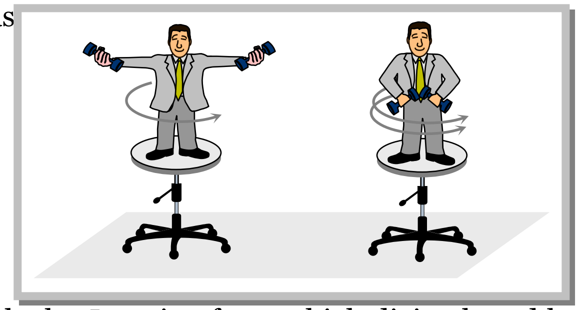
Since angular momentum $I\omega$ remains constant so when I decreases, angular velocity ω increases and vice-versa.

Examples of law of conservation of angular momentum :

(1) The angular velocity of revolution of a planet around the sun in an elliptical orbit increases when the planet come closer to the sun and vice-versa because when planet comes closer to the sun, it's moment of inertia I decreases there fore ω increases.

(2) A circus acrobat performs feats involving spin by bringing his arms and legs closer to his body or vice-versa. On bringing the arms and legs closer to body, his moment of inertia I decreases. Hence ω increases.

(3) A person-carrying heavy weight in his hands and standing on a rotating platform can change the speed of platform. When the person suddenly folds his arms. Its moment of inertia decreases and in accordance the angular speed increas



(4) A diver performs somersaults by Jumping from a high diving board keeping his legs and arms out stretched first and then curling his body.

(5) Effect of change in radius of earth on its time period

Angular momentum of the earth $L = I\omega = \text{constant}$

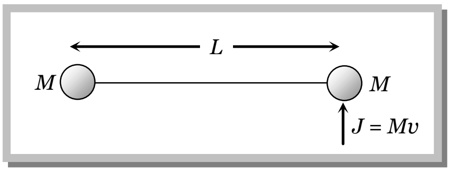
$$L = \frac{2}{5}MR^2 \times \frac{2\pi}{T} = \text{constant}$$

$\therefore$ $T \propto R^2$ [if M remains constant]

If R becomes half then time period will become one-fourth *i.e.* $\frac{24}{4} = 6 hrs.$

Sample problems based on angular momentum

Problem 32. Consider a body, shown in figure, consisting of two identical balls, each of mass M connected by a light rigid rod. If an impulse $J = Mv$ is imparted to the body at one of its ends, what would be its angular velocity **[IIT-JEE (Screening) 2003]**



(a) v/L (b) $2v/L$

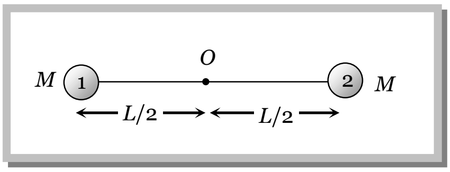
(c) $v/3L$ (d) $v/4L$

Solution: (a) Initial angular momentum of the system about point O

= Linear momentum × Perpendicular distance of linear momentum from the axis of rotation

$= Mv\left(\frac{L}{2}\right)$ ....(i)

Final angular momentum of the system about point O



$= I_1\omega + I_2\omega = (I_1 + I_2)\omega = \left[M\left(\frac{L}{2}\right)^2 + M\left(\frac{L}{2}\right)^2\right]\omega$ ....(ii)

Applying the law of conservation of angular momentum

$\Rightarrow \quad Mv\left(\frac{L}{2}\right) = 2M\left(\frac{L}{2}\right)^2\omega \quad \Rightarrow \quad \omega = \frac{v}{L}$

Problem 33. A thin circular ring of mass M and radius R is rotating about its axis with a constant angular velocity ω. Four objects each of mass m, are kept gently to the opposite ends of two perpendicular diameters of the ring. The angular velocity of the ring will be

(a) $\frac{M\omega}{M+4m}$ (b) $\frac{(M+4m)\omega}{M}$ (c) $\frac{(M-4m)\omega}{M+4m}$ (d) $\frac{M\omega}{4m}$

Solution: (a) Initial angular momentum of ring $= I\omega = MR^2\omega$

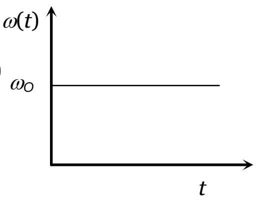
If four object each of mass m, and kept gently to the opposite ends of two perpendicular diameters of the ring then final angular momentum = $(MR^2 + 4mR^2)\omega'$

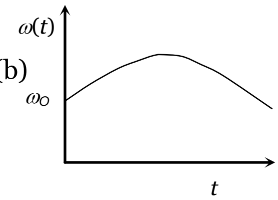
By the conservation of angular momentum

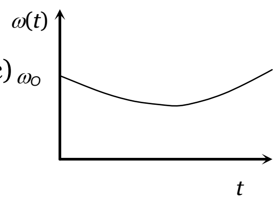
Initial angular momentum = Final angular momentum

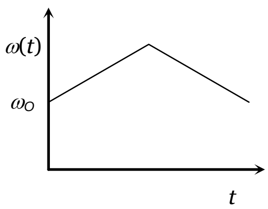
$$MR^2\omega = (MR^2 + 4mR^2)\omega' \Rightarrow \omega' = \left(\frac{M}{M+4m}\right)\omega .$$

Problem 34. A circular platform is free to rotate in a horizontal plane about a vertical axis passing through its center. A tortoise is sitting at the edge of the platform. Now, the platform is given an angular velocity ω_0. When the tortoise moves along a chord of the platform with a constant velocity (with respect to the platform), the angular velocity of the platform $\omega(t)$ will vary with time t as

(a) 

(b) 

(c) 

(d) 

Solution: (b) The angular momentum (L) of the system is conserved *i.e.* $L = I\omega$ = constant

When the tortoise walks along a chord, it first moves closer to the centre and then away from the centre. Hence, M.I. first decreases and then increases. As a result, ω will first increase and then decrease. Also the change in ω will be non-linear function of time.

Problem 35. The position of a particle is given by : $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k})$ and momentum $\vec{P} = (3\hat{i} + 4\hat{j} - 2\hat{k})$. The angular momentum is perpendicular to

(a) X-axis (b) Y-axis

(c) Z-axis (d) Line at equal angles to all the three axes

Solution: (a) $\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = 0\hat{i} - \hat{j} - 2\hat{k} = -\hat{j} - 2\hat{k}$ and the X- axis is given by $i + 0\hat{j} + 0\hat{k}$

Dot product of these two vectors is zero *i.e.* angular momentum is perpendicular to X-axis.

Problem 36. Two discs of moment of inertia I_1 and I_2 and angular speeds ω_1 and ω_2 are rotating along collinear axes passing through their centre of mass and perpendicular to their plane. If the two are made to rotate together along the same axis the rotational KE of system will be

(a) $\frac{I_1\omega_1 + I_2\omega_2}{2(I_1 + I_2)}$ (b) $\frac{(I_1 + I_2)(\omega_1 + \omega_2)^2}{2}$ (c) $\frac{(I_1\omega_1 + I_2\omega_2)^2}{2(I_1 + I_2)}$ (d) None of these

Solution: (c) By the law of conservation of angular momentum $I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$

Angular velocity of system $\omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$

Rotational kinetic energy $= \frac{1}{2}(I_1 + I_2)\omega^2 = \frac{1}{2}(I_1 + I_2)\left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)^2 = \frac{(I_1\omega_1 + I_2\omega_2)^2}{2(I_1 + I_2)}$.

Problem 37. A smooth uniform rod of length L and mass M has two identical beads of negligible size, each of mass m, which can slide freely along the rod. Initially the two beads are at the centre of the rod and the system is rotating with angular velocity ω_0 about an axis perpendicular to the rod and passing through the mid point of the rod (see figure). There are no external forces. When the beads reach the ends of the rod, the angular velocity of the system is

(a) ω_0 (b) $\frac{M\omega_0}{M+12m}$

(c) $\frac{M\omega_0}{M+2m}$ (d) $\frac{M\omega_0}{M+6m}$

Solution: (d) Since there are no external forces therefore the angular momentum of the system remains constant.

Initially when the beads are at the centre of the rod angular momentum $L_1 = \left(\frac{ML^2}{12}\right)\omega_0$ .....(i)

When beads reach the ends of the rod then angular

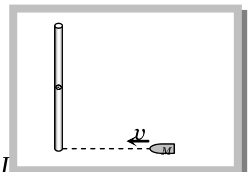
momentum $= \left(m\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2 + \frac{ML^2}{12}\right)\omega'$ ..(ii)

Equating (i) and (ii) $\frac{ML^2}{12}\omega_0 = \left(\frac{mL^2}{2} + \frac{ML^2}{12}\right)\omega' \Rightarrow \omega' = \frac{M\omega_o}{M+6m}$.

***Problem* 38.** Moment of inertia of uniform rod of mass M and length L about an axis through its centre and perpendicular to its length is given by $\frac{ML^2}{12}$. Now consider one such rod pivoted at its centre, free to rotate in a vertical plane. The rod is at rest in the vertical position. A bullet of mass M moving horizontally at a speed v strikes and embedded in one end of the rod. The angular velocity of the rod just after the collision will be

(a) v/L (b) $2v/L$ (c) $3v/2L$ (d) $6v/L$

Solution: (c)Initial angular momentum of the system = Angular momentum of bullet before collision $= Mv\left(\frac{L}{2}\right)$ .....(i)

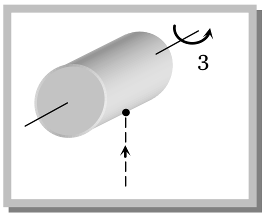


let the rod rotates with angular velocity ω.

Final angular momentum of the system $= \left(\frac{ML^2}{12}\right)\omega + M\left(\frac{L}{2}\right)^2\omega$ ......(ii)

By equation (i) and (ii) $Mv\frac{L}{2} = \left(\frac{ML^2}{12} + \frac{ML^2}{4}\right)\omega$ or $\omega = 3v/2L$

***Problem* 39.** A solid cylinder of mass 2 *kg* and radius $0.2\,m$ is rotating about its own axis without friction with angular velocity $3\,rad/s$. A particle of mass 0.5 *kg* and moving with a velocity 5 *m/s* strikes the cylinder and sticks to it as shown in figure. The angular momentum of the cylinder before collision will be



(a) 0.12 *J-s* (b) 12 *J-s*

(c) 1.2 *J-s* (d) 1.12 *J-s*

Solution: (a) Angular momentum of the cylinder before collision $L = I\omega = \frac{1}{2}MR^2\omega = \frac{1}{2}(2)(0.2)^2 \times 3 = 0.12$ *J-s*.

***Problem* 40.** In the above problem the angular velocity of the system after the particle sticks to it will be

(a) 0.3 *rad/s* (b) 5.3 *rad/s* (c) 10.3 *rad/s* (d) 89.3 *rad/s*

Solution: (c) Initial angular momentum of bullet + initial angular momentum of cylinder

= Final angular momentum of (bullet + cylinder) system

$\Rightarrow mvr + I_1\omega = (I_1 + I_2)\omega'$

$\Rightarrow mvr + I_1\omega = \left(\frac{1}{2}Mr^2 + mr^2\right)\omega'$

$\Rightarrow 0.5 \times 5 \times 0.2 + 0.12 = \left(\frac{1}{2}2(0.2)^2 + (0.5)(0.2)^2\right)\omega'$

$\therefore \omega' = 10.3\ rad/sec.$

7.19 Work, Energy and Power for Rotating Body

(1) **Work :** If the body is initially at rest and angular displacement is $d\theta$ due to torque then work done on the body.

$W = \int \tau\, d\theta$ [Analogue to work in translatory motion $W = \int F\, dx$]

(2) **Kinetic energy :** The energy, which a body has by virtue of its rotational motion is called rotational kinetic energy. A body rotating about a fixed axis possesses kinetic energy because its constituent particles are in motion, even though the body as a whole remains in place.

Rotational kinetic energy	Analogue to translatory kinetic energy
$K_R = \frac{1}{2}I\omega^2$	$K_T = \frac{1}{2}mv^2$
$K_R = \frac{1}{2}L\omega$	$K_T = \frac{1}{2}Pv$
$K_R = \frac{L^2}{2I}$	$K_T = \frac{P^2}{2m}$

(3) **Power :** Rate of change of kinetic energy is defined as power

$$P = \frac{d}{dt}(K_R) = \frac{d}{dt}\left[\frac{1}{2}I\omega^2\right] = I\omega\frac{d\omega}{dt} = I\omega\alpha = I\alpha\omega = \tau\omega$$

In vector form Power $= \vec{\tau}\cdot\vec{\omega}$ [Analogue to power in translatory motion $P = \vec{F}\cdot\vec{v}$]

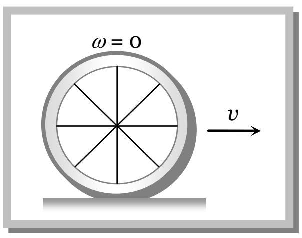
7.20 Slipping, Spinning and Rolling

(1) **Slipping :** When the body slides on a surface without rotation then its motion is called slipping motion.

In this condition friction between the body and surface $F = 0$.

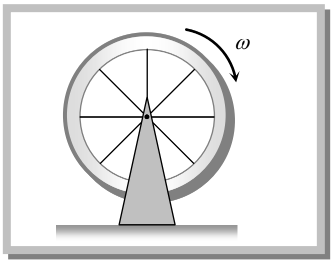
Body possess only translatory kinetic energy $K_T = \frac{1}{2}mv^2$.

Example : Motion of a ball on a frictionless surface.



(2) **Spinning :** When the body rotates in such a manner that its axis of rotation does not move then its motion is called spinning motion.

In this condition axis of rotation of a body is fixed.



Example : Motion of blades of a fan.

In spinning, body possess only rotatory kinetic energy $K_R = \frac{1}{2} I\omega^2$.

or $$K_R = \frac{1}{2} mK^2 \frac{v^2}{R^2} = \frac{1}{2} mv^2 \left(\frac{K^2}{R^2}\right)$$

i.e., Rotatory kinetic energy = $\left(\frac{K^2}{R^2}\right)$ times translatory kinetic energy.

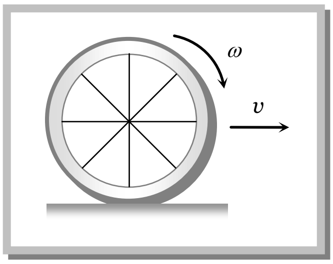
Here $\frac{K^2}{R^2}$ is a constant for different bodies. Value of $\frac{K^2}{R^2} = 1$ (ring), $\frac{K^2}{R^2} = \frac{1}{2}$ (disc) and $\frac{K^2}{R^2} = \frac{1}{2}$ (solid sphere)

(3) **Rolling :** If in case of rotational motion of a body about a fixed axis, the axis of rotation also moves, the motion is called combined translatory and rotatory.

Example : (i) Motion of a wheel of cycle on a road.

(ii) Motion of football rolling on a surface.

In this condition friction between the body and surface $F \neq 0$.



Body possesses both translational and rotational kinetic energy.

Net kinetic energy = (Translatory + Rotatory) kinetic energy.

$$K_N = K_T + K_R = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} mv^2 + \frac{1}{2} mv^2 \frac{K^2}{R^2}$$

$$\therefore \quad K_N = \frac{1}{2} mv^2 \left(1 + \frac{K^2}{R^2}\right)$$

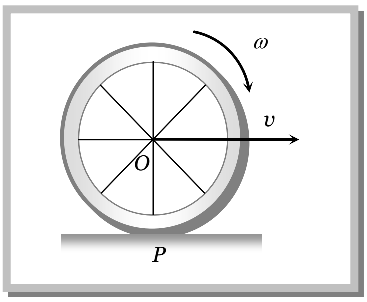
7.21 Rolling Without Slipping

In case of combined translatory and rotatory motion if the object rolls across a surface in such a way that there is no relative motion of object and surface at the point of contact, the motion is called rolling without slipping.

Friction is responsible for this type of motion but work done or dissipation of energy against friction is zero as there is no relative motion between body and surface at the point of contact.

Rolling motion of a body may be treated as a pure rotation about an axis through point of contact with same angular velocity ω.

By the law of conservation of energy



$$K_N = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2 \qquad [\therefore \text{ As } v = R\omega]$$

$$= \frac{1}{2} mR^2\omega^2 + \frac{1}{2} I\omega^2$$

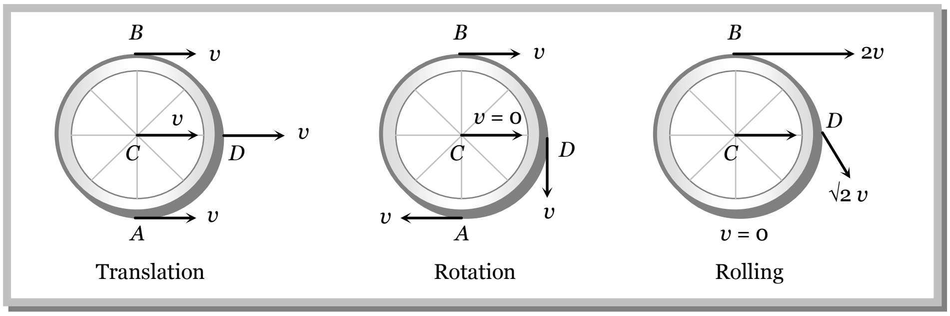
$$= \frac{1}{2} \omega^2 [mR^2 + I]$$

$$= \frac{1}{2}\omega^2[I + mR^2] = \frac{1}{2}I_P\omega^2 \quad [\text{As } I_P = I + mR^2]$$

By theorem of parallel axis, where I = moment of inertia of rolling body about its centre '*O*' and I_P = moment of inertia of rolling body about point of contact '*P*'.

(1) **Linear velocity of different points in rolling :** In case of rolling, all points of a rigid body have same angular speed but different linear speed.

Let *A, B, C* and *D* are four points then their velocities are shown in the following figure.



(2) **Energy distribution table for different rolling bodies :**

Body	$\frac{K^2}{R^2}$	**Translatory (K_T)** $\frac{1}{2}mv^2$	**Rotatory (K_R)** $\frac{1}{2}mv^2\frac{K^2}{R^2}$	**Total (K_N)** $\frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right)$	$\frac{K_T}{K_N}$ **(%)**	$\frac{K_R}{K_N}$ **(%)**
Ring Cylindrical shell	1	$\frac{1}{2}mv^2$	$\frac{1}{2}mv^2$	mv^2	$\frac{1}{2}$ (50%)	$\frac{1}{2}$ (50%)
Disc solid cylinder	$\frac{1}{2}$	$\frac{1}{2}mv^2$	$\frac{1}{4}mv^2$	$\frac{3}{4}mv^2$	$\frac{2}{3}$ (66.6%)	$\frac{1}{3}$ (33.3%)
Solid sphere	$\frac{2}{5}$	$\frac{1}{2}mv^2$	$\frac{1}{5}mv^2$	$\frac{7}{10}mv^2$	$\frac{5}{7}$ (71.5%)	$\frac{2}{7}$ (28.5%)
Hollow sphere	$\frac{2}{3}$	$\frac{1}{2}mv^2$	$\frac{1}{3}mv^2$	$\frac{5}{6}mv^2$	$\frac{3}{5}$ (60%)	$\frac{2}{5}$ (40%)

Sample problems based on kinetic energy, work and power

***Problem* 41.** A ring of radius 0.5 *m* and mass 10 *kg* is rotating about its diameter with an angular velocity of 20 *rad/s*. Its kinetic energy is

(a) 10 *J* (b) 100 *J* (c) 500 *J* (d) 250 *J*

Solution: (d) Rotational kinetic energy $\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 = \frac{1}{2}\left(\frac{1}{2}\times 10\times(0.5)^2\right)(20)^2 = 250\ J$

***Problem* 42.** An automobile engine develops 100 *kW* when rotating at a speed of 1800 rev/*min*. What torque does it deliver **[CBSE PMT 2000]**

(a) 350 *N-m* (b) 440 *N-m* (c) 531 *N-m* (d) 628 *N-m*

Solution: (c) $P = \tau\omega \Rightarrow \tau = \dfrac{100\times 10^3}{2\pi\dfrac{1800}{60}} = 531\ N\text{-}m$

***Problem* 43.** A body of moment of inertia of 3 *kg-m*² rotating with an angular velocity of 2 *rad/sec* has the same kinetic energy as a mass of 12 *kg* moving with a velocity of

(a) 8 *m/s* (b) 0.5 *m/s* (c) 2 *m/s* (d) 1 *m/s*

Solution: (d) Rotational kinetic energy of the body = $\frac{1}{2}I\omega^2$ and translatory kinetic energy $= \frac{1}{2}mv^2$

According to problem $= \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 \Rightarrow \frac{1}{2}\times 3\times(2)^2 = \frac{1}{2}\times 12\times v^2 \Rightarrow v = 1\, m/s$.

***Problem* 44.** A disc and a ring of same mass are rolling and if their kinetic energies are equal, then the ratio of their velocities will be

(a) $\sqrt{4}:\sqrt{3}$ (b) $\sqrt{3}:\sqrt{4}$ (c) $\sqrt{3}:\sqrt{2}$ (d) $\sqrt{2}:\sqrt{3}$

Solution: (a) $K_{disc} = \frac{1}{2}mv_d^2\left(1+\frac{k^2}{R^2}\right) = \frac{3}{4}mv_d^2$ $\left[As\ \frac{k^2}{R^2} = \frac{1}{2}\ \text{for disc}\right]$

$K_{ring} = \frac{1}{2}mv_r\left(1+\frac{k^2}{R^2}\right) = mv_r^2$ $\left[As\ \frac{k^2}{R^2} = 1\ \text{for ring}\right]$

According to problem $K_{disc} = K_{ring} \Rightarrow \frac{3}{4}mv_d^2 = mv_r^2 \Rightarrow \frac{v_d}{v_r} = \sqrt{\frac{4}{3}}$.

***Problem* 45.** A wheel is rotating with an angular speed of $20\, rad/sec$. It is stopped to rest by applying a constant torque in $4s$. If the moment of inertia of the wheel about its axis is 0.20 *kg-m*², then the work done by the torque in two seconds will be

(a) 10 *J* (b) 20 *J* (c) 30 *J* (d) 40 *J*

Solution: (c) $\omega_1 = 20\ rad/sec,\ \omega_2 = 0, t = 4 sec$. So angular retardation $\alpha = \frac{\omega_1-\omega_2}{t} = \frac{20}{4} = 5 rad/sec^2$

Now angular speed after 2 *sec* $\omega_2 = \omega_1 - \alpha t = 20 - 5\times 2 = 10\ rad/sec$

Work done by torque in 2 *sec* = loss in kinetic energy $= \frac{1}{2}I\left(\omega_1^2-\omega_2^2\right) = \frac{1}{2}(0.20)((20)^2-(10)^2)$

$= \frac{1}{2}\times 0.2\times 300 = 30\, J.$

***Problem* 46.** If the angular momentum of a rotating body is increased by 200%, then its kinetic energy of rotation will be increased by

(a) 400% (b) 800% (c) 200% (d) 100%

Solution: (b) As $E = \frac{L^2}{2I} \Rightarrow \frac{E_2}{E_1} = \left(\frac{L_2}{L_1}\right)^2 = \left(\frac{3L_1}{L_1}\right)^2$ [As $L_2 = L_1 + 200\%.L_1 = 3L_1$]

$\Rightarrow E_2 = 9E_1 = E_1 + 800\%\ \text{of}\ E_1$

***Problem* 47.** A ring, a solid sphere and a thin disc of different masses rotate with the same kinetic energy. Equal torques are applied to stop them. Which will make the least number of rotations before coming to rest

(a) Disc (b) Ring

(c) Solid sphere (d) All will make same number of rotations

Solution: (d) As $W = \tau\theta$ = Energy $\Rightarrow \theta = \frac{\text{Energy}}{\tau} = 2n\pi$

So, if energy and torque are same then all the bodies will make same number of rotation.

***Problem* 48.** The angular velocity of a body is $\vec{\omega} = 2\hat{i}+3\hat{j}+4\hat{k}$ and a torque $\vec{\tau} = \hat{i}+2\hat{j}+3\hat{k}$ acts on it. The rotational power will be

(a) 20 *W* (b) 15 *W* (c) $\sqrt{17}$ *W* (d) $\sqrt{14}$ *W*

Solution: (a) Power $(P) = \vec{\tau}.\vec{\omega} = (i+2\hat{j}+3\hat{k}).(2\hat{i}+3\hat{j}+4\hat{k}) = 2+6+12 = 20\, W$

***Problem* 49.** A flywheel of moment of inertia 0.32 *kg-m*² is rotated steadily at $120\, rad/\sec$ by a $50\, W$ electric motor. The kinetic energy of the flywheel is

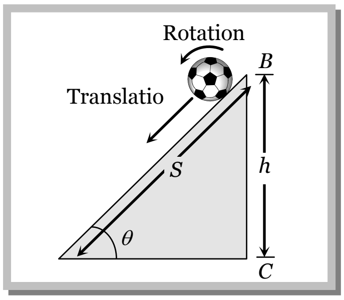
(a) 4608 *J* (b) 1152 *J* (c) 2304 *J* (d) 6912 *J*

Solution: (c) Kinetic energy $K_R = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.32)(120)^2 = 2304\, J.$

7.22 Rolling on an Inclined Plane

When a body of mass m and radius R rolls down on inclined plane of height 'h' and angle of inclination θ, it loses potential energy. However it acquires both linear and angular speeds and hence, gain kinetic energy of translation and that of rotation.

By conservation of mechanical energy $mgh = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$



(1) **Velocity at the lowest point :** $v = \sqrt{\dfrac{2gh}{1+\dfrac{k^2}{R^2}}}$

(2) **Acceleration in motion :** From equation $v^2 = u^2 + 2aS$

By substituting $u = 0$, $S = \dfrac{h}{\sin\theta}$ and $v = \sqrt{\dfrac{2gh}{1+\dfrac{k^2}{R^2}}}$ we get

$$a = \frac{g\sin\theta}{1+\dfrac{k^2}{R^2}}$$

(3) **Time of descent :** From equation $v = u + at$

By substituting u = 0 and value of v and a from above expressions

$$t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left[1+\frac{k^2}{R^2}\right]}$$

From the above expressions it is clear that, $v \propto \dfrac{1}{\sqrt{1+\dfrac{k^2}{R^2}}}$; $a \propto \dfrac{1}{1+\dfrac{k^2}{R^2}}$; $t \propto \sqrt{1+\dfrac{k^2}{R^2}}$

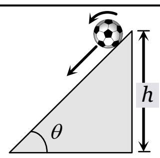
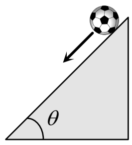
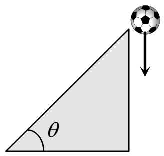
Note :
- Here factor $\left(\dfrac{k^2}{R^2}\right)$ is a measure of moment of inertia of a body and its value is constant for given shape of the body and it does not depend on the mass and radius of a body.
- Velocity, acceleration and time of descent (for a given inclined plane) all depends on $\dfrac{k^2}{R^2}$.

  Lesser the moment of inertia of the rolling body lesser will be the value of $\dfrac{k^2}{R^2}$. So greater will be its velocity and acceleration and lesser will be the time of descent.
- If a solid and hollow body of same shape are allowed to roll down on inclined plane then as $\left(\dfrac{k^2}{R^2}\right)_S < \left(\dfrac{k^2}{R^2}\right)_H$, solid body will reach the bottom first with greater velocity.
- If a ring, cylinder, disc and sphere runs a race by rolling on an inclined plane then as $\left(\dfrac{k^2}{R^2}\right)_{\text{sphere}}$ = minimum while $\left(\dfrac{k^2}{R^2}\right)_{\text{Ring}}$ = maximum , the sphere will reach the bottom first with greatest velocity while ring at last with least velocity.
- Angle of inclination has no effect on velocity, but time of descent and acceleration depends on it.

  velocity $\propto \theta^{\circ}$, time of decent $\propto \theta^{-1}$ and acceleration $\propto \theta$.

7.23 Rolling Sliding and Falling of a Body

	Figure	Velocity	Acceleration	Time

Rolling	$\frac{k^2}{R^2} \neq 0$		$\sqrt{\frac{2gh}{1+k^2/R^2}}$	$\frac{g\sin\theta}{1+K^2/R^2}$	$\frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{k^2}{R^2}\right)}$
Sliding	$\frac{k^2}{R^2} = 0$		$\sqrt{2gh}$	$g\sin\theta$	$\frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}$
Falling	$\frac{k^2}{R^2} = 0$ $\theta = 90^o$		$\sqrt{2gh}$	g	$\sqrt{\frac{2h}{g}}$

7.24 Velocity, Acceleration and Time for Different Bodies

Body	$\frac{k^2}{R^2}$	Velocity $v = \sqrt{\frac{2gh}{1+\frac{k^2}{R^2}}}$	Acceleration $a = \frac{g\sin\theta}{1+\frac{k^2}{R^2}}$	Time of descent $t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{k^2}{R^2}\right)}$
Ring or Hollow cylinder	1	$\sqrt{gh}$	$\frac{1}{2}g\sin\theta$	$\frac{1}{\sin\theta}\sqrt{\frac{4h}{g}}$
Disc or solid cylinder	$\frac{1}{2}$ or 0.5	$\sqrt{\frac{4gh}{3}}$	$\frac{2}{3}g\sin\theta$	$\frac{1}{\sin\theta}\sqrt{\frac{3h}{g}}$
Solid sphere	$\frac{2}{5}$ or 0.4	$\sqrt{\frac{10}{7}gh}$	$\frac{5}{7}g\sin\theta$	$\frac{1}{\sin\theta}\sqrt{\frac{14}{5}\frac{h}{g}}$
Hollow sphere	$\frac{2}{3}$ or 0.66	$\sqrt{\frac{6}{5}gh}$	$\frac{3}{5}g\sin\theta$	$\frac{1}{\sin\theta}\sqrt{\frac{10}{3}\frac{h}{g}}$

Sample problems based on rolling on an inclined plane

Problem 50. A solid cylinder of mass M and radius R rolls without slipping down an inclined plane of length L and height h. What is the speed of its centre of mass when the cylinder reaches its bottom

(a) $\sqrt{\frac{3}{4}gh}$ (b) $\sqrt{\frac{4}{3}gh}$ (c) $\sqrt{4gh}$ (d) $\sqrt{2gh}$

Solution: (b) Velocity at the bottom $(v) = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}} = \sqrt{\frac{2gh}{1+\frac{1}{2}}} = \sqrt{\frac{4}{3}gh}$.

Problem 51. A sphere rolls down on an inclined plane of inclination θ. What is the acceleration as the sphere reaches bottom **[Orissa JEE 2003]**

(a) $\frac{5}{7}g\sin\theta$ (b) $\frac{3}{5}g\sin\theta$ (c) $\frac{2}{7}g\sin\theta$ (d) $\frac{2}{5}g\sin\theta$

Solution: (a) Acceleration $(a) = \frac{g\sin\theta}{1+\frac{K^2}{R^2}} = \frac{g\sin\theta}{1+\frac{2}{5}} = \frac{5}{7}g\sin\theta$.

Problem 52. A ring solid sphere and a disc are rolling down from the top of the same height, then the sequence to reach on surface is **[RPMT 1999]**

(a) Ring, disc, sphere (b) Sphere, disc, ring (c) Disc, ring, sphere (d) Sphere, ring, disc

Solution: (b) Time of descent $\propto$ moment of inertia $\propto \frac{k^2}{R^2}$

$$\left(\frac{k^2}{R^2}\right)_{sphere} = 0.4\,,\ \left(\frac{k^2}{R^2}\right)_{disc} = 0.5\,,\ \left(\frac{k^2}{R^2}\right)_{ring} = 1 \quad \therefore\ t_{sphere} < t_{disc} < t_{ring}.$$

Problem **53.** A thin uniform circular ring is rolling down an inclined plane of inclination 30° without slipping. Its linear acceleration along the inclined plane will be

(a) $g/2$ (b) $g/3$ (c) $g/4$ (d) $2g/3$

Solution: (c) $a = \dfrac{g\sin\theta}{1+\frac{k^2}{R^2}} = \dfrac{g\sin 30^o}{1+1} = \dfrac{g}{4}$ [As $\frac{k^2}{R^2} = 1$ and $\theta = 30^o$]

Problem **54.** A solid sphere and a disc of same mass and radius starts rolling down a rough inclined plane, from the same height the ratio of the time taken in the two cases is

(a) 15 : 14 (b) $\sqrt{15}:\sqrt{14}$ (c) 14 : 15 (d) $\sqrt{14}:\sqrt{15}$

Solution: (d) Time of descent $t = \dfrac{1}{\sin\theta}\sqrt{\dfrac{2h}{g}\left(1+\dfrac{k^2}{R^2}\right)}$ $\therefore \dfrac{t_{\text{shpere}}}{t_{\text{disc}}} = \sqrt{\dfrac{\left(1+\frac{k^2}{R^2}\right)_{\text{sphere}}}{\left(1+\frac{k^2}{R^2}\right)_{\text{disc}}}} = \sqrt{\dfrac{1+\frac{2}{5}}{1+\frac{1}{2}}} = \sqrt{\dfrac{7}{5}\times\dfrac{2}{3}} = \sqrt{\dfrac{14}{15}}$

Problem **55.** A solid sphere of mass $0.1\,kg$ and radius $2\,cm$ rolls down an inclined plane $1.4\,m$ in length (slope 1 in 10). Starting from rest its final velocity will be

(a) $1.4\ m/\text{sec}$ (b) $0.14\ m/\text{sec}$ (c) $14\ m/\text{sec}$ (d) $0.7\ m/\text{sec}$

Solution: (a) $v = \sqrt{\dfrac{2gh}{1+\frac{k^2}{R^2}}} = \sqrt{\dfrac{2\times 9.8\times l\sin\theta}{1+\frac{2}{5}}}$ [As $\frac{k^2}{R^2} = \frac{2}{5}$, $l = \frac{h}{\sin\theta}$ and $\sin\theta = \frac{1}{10}$ given]

$$\Rightarrow v = \sqrt{\frac{2\times 9.8\times 1.4\times\frac{1}{10}}{7/5}} = 1.4\,m/s.$$

Problem **56.** A solid sphere rolls down an inclined plane and its velocity at the bottom is v_1. Then same sphere slides down the plane (without friction) and let its velocity at the bottom be v_2. Which of the following relation is correct

(a) $v_1 = v_2$ (b) $v_1 = \frac{5}{7}v_2$ (c) $v_1 = \frac{7}{5}v_2$ (d) None of these

Solution: (d) When solid sphere rolls down an inclined plane the velocity at bottom $v_1 = \sqrt{\frac{10}{7}gh}$

but, if there is no friction then it slides on inclined plane and the velocity at bottom $v_2 = \sqrt{2gh}$

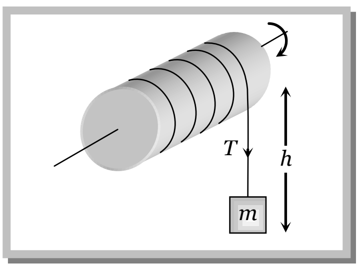
$$\therefore \frac{v_1}{v_2} = \sqrt{\frac{5}{7}}.$$

7.25 Motion of Connected Mass

A point mass is tied to one end of a string which is wound round the solid body [cylinder, pulley, disc]. When the mass is released, it falls vertically downwards and the solid body rotates unwinding the string

m = mass of point-mass, M = mass of a rigid body

R = radius of a rigid body, I = moment of inertia of rotating body



(1) **Downwards acceleration of point mass** $a = \dfrac{g}{1+\frac{I}{mR^2}}$

(2) **Tension in string** $T = mg\left[\dfrac{I}{I+mR^2}\right]$

(3) **Velocity of point mass** $v = \sqrt{\dfrac{2gh}{1+\dfrac{I}{mR^2}}}$ (4) **Angular velocity of rigid body** $\omega = \sqrt{\dfrac{2mgh}{I+mR^2}}$

Sample problems based on motion of connected mass

Problem 57. A cord is wound round the circumference of wheel of radius *r*. The axis of the wheel is horizontal and moment of inertia about it is *I*. A weight *mg* is attached to the end of the cord and falls from rest. After falling through a distance *h,* the angular velocity of the wheel will be

(a) $\sqrt{\dfrac{2gh}{I+mr}}$ (b) $\sqrt{\dfrac{2mgh}{I+mr^2}}$ (c) $\sqrt{\dfrac{2mgh}{I+2mr^2}}$ (d) $\sqrt{2gh}$

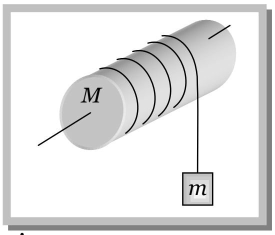
Solution : (b) According to law of conservation of energy $mgh = \frac{1}{2}(I+mr^2)\omega^2 \Rightarrow \omega = \sqrt{\dfrac{2mgh}{I+mr^2}}$.

Problem 58. In the following figure, a body of mass *m* is tied at one end of a light string and this string is wrapped around the solid cylinder of mass *M* and radius *R*. At the moment *t* = 0 the system starts moving. If the friction is negligible, angular velocity at time t would be

(a) $\dfrac{mgRt}{(M+m)}$ (b) $\dfrac{2Mgt}{(M+2m)}$

(c) $\dfrac{2mgt}{R(M-2m)}$ (d) $\dfrac{2mgt}{R(M+2m)}$

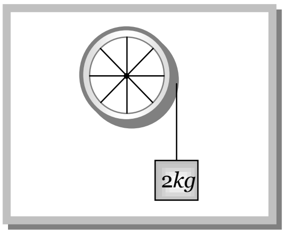
Solution : (d) We know the tangential acceleration $a = \dfrac{g}{1+\dfrac{I}{mR^2}} = \dfrac{g}{1+\dfrac{1/2MR^2}{mR^2}} = \dfrac{2mg}{2m+M}$ [As $I = \frac{1}{2}MR^2$ for cylinder]



After time *t*, linear velocity of mass *m*, $v = u + at = 0 + \dfrac{2mgt}{2m+M}$

So angular velocity of the cylinder $\omega = \dfrac{v}{R} = \dfrac{2mgt}{R(M+2m)}$.

Problem 59. A block of mass $2\,kg$ hangs from the rim of a wheel of radius $0.5\,m$. On releasing from rest the block falls through $5\,m$ height in $2\,s$. The moment of inertia of the wheel will be



(a) 1 $kg\text{-}m^2$ (b) 3.2 $kg\text{-}m^2$
(c) 2.5 $kg\text{-}m^2$ (d) 1.5 $kg\text{-}m^2$

Solution : (d) On releasing from rest the block falls through 5*m* height in 2 *sec*.

$5 = 0 + \frac{1}{2}a(2)^2$ [As $S = ut + \frac{1}{2}at^2$] $\therefore$ $a = 2.5\,m/s^2$

Substituting the value of *a* in the formula $a = \dfrac{g}{1+\dfrac{I}{mR^2}}$ and by solving we get

$$\Rightarrow \quad 2.5 = \frac{10}{1+\dfrac{I}{2\times(0.5)^2}} \Rightarrow I = 1.5 kg - m^2$$